Real Value Added: Incidence and Comovement

Alejandro Rojas-Bernal

University of Hawai'i at Manoa, Department of Economics

rojasber@hawaii.edu

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Abstract

I develop a theory of real value-added incidence for arbitrary firm clusters: sectors, regions, countries, supply chains, or nongeographic groups of firms. In a general-equilibrium economy with input-output linkages, heterogeneous households, and rent-generating wedges, I show that cluster real-value-added growth decomposes into a factor-supply component and total factor productivity (TFP), with TFP further split into technology, competitiveness, and distributional reallocation. The sufficient statistics generalize Domar weights from aggregate productivity accounting to incidence accounting: they measure how firm shocks, factor supplies, household expenditure, firm revenue, markups, ownership, and network structure affect the real value added of a given cluster. I implement the decomposition for U.S. sectors. Aggregate TFP is shaped by a small set of sectoral contributors and drags: computers and electronics and oil and gas extraction are major positive contributors, while farms, construction, and housing are major drags. Sectoral TFP comovement is distinct from level contribution: large Domar sectors and upstream cost-central sectors tend to move against the rest of the economy, while sectors with concentrated downstream customer bases are less synchronized with the aggregate. Around the Great Recession, the TFP measure reveals pre-crisis expansion without efficiency gains and crisis-period downsizing with rising measured efficiency, patterns not visible in revenue data; it also identifies selective medium-run scarring.

1 Introduction

Where is real value added created, and how does it change when shocks hit different parts of the economy? Growth accounting answers this for the aggregate economy. Since Solow (1957), productivity has been measured as the residual part of output growth not explained by share-weighted input growth. Domar (1961) showed that, once intermediate inputs and industries are introduced, the aggregation of these residuals is itself an economic problem. Hulten (1978) provided the classical solution: in an efficient economy, the first-order effect of a sectoral productivity shock on aggregate productivity is the sector's Domar weight. This result underlies modern productivity accounting and much of the empirical practice that links sectoral productivity to aggregate output.

Yet the aggregate economy is rarely the only object of interest. Policymakers ask whether a shock raises real GDP in one country or another. Firms ask whether disruptions to suppliers change value added in their industry. Researchers ask whether technology shocks in one sector propagate to other sectors and comove with productivity in the rest of the economy. Such questions are not aggregate questions; they are incidence questions. They ask how a shock changes real value added for a specific subset of firms: a sector, a region, a country, a supply chain, or a group of firms exposed to the same factor market or demand source. I call such a subset a *firm cluster*. This paper extends Domar–Hulten accounting from aggregate productivity to real-value-added incidence for arbitrary firm clusters.

The distinction matters because value added is not the same object as sales, output, income, welfare, or consumption. For a firm cluster, nominal value added can be written as revenue net of intermediate input expenditure, or equivalently as the sum of primary-factor payments and rents generated by firms in the cluster. These two representations coincide as accounting identities, but they separate different economic margins. A productivity shock to a supplier outside the cluster may lower the cluster's intermediate input costs. A markup shock inside the cluster may change rents and revenue. A transfer across households may change the composition of final demand, shifting production toward or away from the cluster. Thus, real value added for a cluster is not pinned down only by the technology of firms inside the cluster. It depends on the production network, wedges, factor use, ownership, and household expenditure.

To capture these margins, I use a static, nonparametric general-equilibrium economy with constant returns to scale. Firms trade intermediate inputs and hire primary factors; households differ in factor supplies, preferences, and ownership claims; and firms face rent-generating price-cost wedges, with profits rebated to households. I impose no cluster-level factor-market segmentation, allowing factors to relocate across clusters.

The main result is a first-order decomposition of cluster-level real-value-added growth. For any partition $\{N_r\}_{r\in\mathscr{R}}$ of firms, I write cluster r's real value added as $Y_r = TFP_r\, F_r\left(\{L_h\}_{h\in\mathscr{H}}\right)$, where F_r is the cluster-specific aggregator over factors implied by the equilibrium allocation. Theorem 1 decomposes $d\log Y_r$ into four components. The first is a technology component: productivity shocks anywhere in the economy affect cluster r through cluster-specific rent-adjusted Domar weights. The second is a factor-supply channel: changes in factor supplies affect Y_r through cluster-specific factor-incidence weights. The third is a wedge component: changes in markups affect cluster real value added through both direct rent effects and induced reallocations. The fourth is a distributional component: changes in household expenditure and budget shares, factor-income shares, and firm input-cost shares affect Y_r by reallocating demand and resources across firms with different cluster-specific marginal values.

The sufficient statistics in Theorem 1 are the paper's central objects. They generalize Domar weights from aggregate to incidence accounting. Firm-incidence and factor-incidence weights measure the effects of firm shocks and primary-factor supplies on a cluster's real value added. Expenditure and revenue centralities map household spending and firm sales into the factor margins most relevant for the cluster. They answer: *How important is this firm, factor, or household for a cluster's real value added?*

A useful benchmark clarifies the relationship to Hulten's theorem. Suppose all firms in cluster r are competitive, and factor markets are cluster-specific, so cluster value added consists only of payments to factors. Then all rent and demand-reallocation terms vanish, and $d\log Y_r = \sum_{i\in\mathscr{N}_r}\frac{\lambda_i}{\Phi_r}\,d\log A_i \sum_{h\in\mathscr{H}_r}\frac{\Lambda_h}{\Phi_r}\,d\log L_h$, where λ_i is firm i's Domar weight, A_i is firm-level productivity, Λ_h is factor h's income share, Φ_r is cluster r's value-added share, and L_h is factor supply. Thus, cluster-level real value-added growth reduces to a weighted average of within-cluster productivity and factor-supply growth. This is a cluster-specific Hulten theorem. It does not require the entire economy to be closed or efficient; it requires only that the cluster itself generate no rents and factor market segmentation. Away from this knife-edge benchmark, demand composition, ownership, markups, and reallocation become first-order determinants of Y_r.

This benchmark also clarifies the relationship with open-economy real GDP accounting. Burstein & Cravino (2015) and Baqaee & Farhi (2024) extend Hulten-style accounting to open economies. Baqaee & Farhi (2024), in particular, characterize real GDP and welfare for countries with wedge-like distortions. Their country-level real GDP decomposition is a special case of the cluster problem: it applies when the cluster is a country and factor markets are country-specific. Under those restrictions, external exposure is represented using variations in net imported quantities. My results remove these restrictions and apply to any firm cluster, while allowing factors to relocate across cluster boundaries. Additionally, the sufficient statistics differ; rather than requiring changes in net imported quantities, the decomposition is expressed in terms of cluster-specific incidence weights and nominal shares. This matters empirically because imported quantities are rarely observed in input-output data and typically require additional price information. Theorem 1 therefore nests the country-level real GDP case while providing an implementable decomposition for general cluster settings.

The empirical application constructs U.S. sector-level TFP growth over 1997–2021. The first exercise measures cumulative sectoral contributions to aggregate TFP and identifies computers and electronics and oil and gas extraction as the largest positive contributors; farms, construction, and housing are the largest drags.

The second exercise studies comovement. I compute each sector's TFP correlation with the TFP for the rest of the economy. The most positively comoving sectors are discretionary services and transportation, while the most negatively comoving sectors are farms, housing, and financial sectors. I then relate comovement to network position. Larger sectors, sectors through which a larger share of aggregate value added passes, and more upstream sectors tend to move against the rest of the economy. Sectors with concentrated downstream customer bases also decouple from the aggregate: their TFP growth loads more on buyer-specific shocks than on broad economy-wide movements.

The third exercise studies sectoral TFP around the Great Financial Crisis. The literature emphasizes that aggregate labor productivity and TFP slowed before the Great Recession, with the slowdown concentrated in IT-producing and IT-intensive sectors rather than housing or finance (Fernald, 2015; Fernald et al., 2017; Syverson, 2017). My object is different: rather than explaining the aggregate productivity slowdown, I ask which sectors experienced TFP contraction, recovery, and medium-run scarring.

The results show that the GFC was not a uniform sectoral TFP collapse. Some sectors experienced large TFP declines before 2008; others contracted during the core 2008–2009 recession; and the sectors with the largest short-run losses were not necessarily those with the largest medium-run scars. The sectoral TFP estimates also contain information that revenue alone would miss. In the pre-crisis stress period, several crisis-relevant sectors expanded in revenue while measured TFP declined, including credit intermediation, securities and investments, funds and trusts, and construction. This pattern is consistent with expansion without corresponding efficiency gains. During the core crisis, the reverse pattern appears in some financial sectors: revenue falls while measured TFP rises, as in securities and investments and credit intermediation, consistent with downsizing, selection, or the exit of low-productivity activity. Additionally, sectors showed a strong bounce-back from the 2008–2009 decline, and medium-run scarring was selective. Sectors with more concentrated customer bases exhibit smaller scars, whereas goods-producing sectors remain substantially more scarred.

The final empirical exercise compares partial-equilibrium sufficient statistics with full general-equilibrium responses to sectoral productivity and markup shocks. It asks when local statistics provide reliable back-of-the-envelope approximations to full equilibrium effects. For productivity shocks, the sufficient statistics closely track the equilibrium response; for markup shocks, the approximation is much weaker.

The paper contributes to several literatures. First, it contributes to productivity and growth accounting (Solow, 1957; Domar, 1961; Hulten, 1978; Jorgenson et al., 1987; Hall, 1988; Basu & Fernald, 2002; Petrin & Levinsohn, 2012). This literature maps technology, wedges, and reallocation into aggregate productivity. I change both the unit and object of measurement: instead of recovering an aggregate residual, I derive an analytical incidence decomposition for real value added in arbitrary firm clusters.

Second, I contribute to the literature on misallocation and distortions (Harberger, 1964; Restuccia & Rogerson, 2008; Hsieh & Klenow, 2009; Jones, 2011; Loecker & Warzynski, 2012; De Loecker et al., 2020; Liu, 2019; Bigio & La'O, 2020; Baqaee & Farhi, 2020; Rojas-Bernal, 2026). This literature shows that wedges, market power, and misallocation can make measured TFP differ from technology, and that input-output linkages shape the aggregate cost of distortions. I characterize how rent-generating wedges change real value added for arbitrary firm clusters.

Third, I contribute to the literature on micro shocks, sectoral shocks, and comovement (Horvath, 1998, 2000; Dupor, 1999; Gabaix, 2011; Foerster et al., 2011; Acemoglu et al., 2012; Carvalho et al., 2021; Huo et al., 2025). This literature studies when idiosyncratic shocks survive aggregation through firm granularity and input-output propagation. I study a complementary object. Rather than decomposing aggregate volatility or international comovement, I estimate sectoral real-value-added TFP, its contribution to aggregate TFP levels, its comovement, and cross-sectoral propagation.

Fourth, I contribute to open-economy value-added and real-output accounting (Diewert & Morrison, 1986; Kohli, 2004; Kehoe & Ruhl, 2008; Johnson & Noguera, 2012; Koopman et al., 2014; Burstein & Cravino, 2015; Los et al., 2016; Devereux et al., 2023). This literature separates real output from real income when terms of trade move, distinguishing productivity from trade-cost and external-price effects. Rather than tracing value added across countries or measuring country-level GDP responses to trade and fiscal shocks, I characterize real-value-added incidence for arbitrary firm clusters with analytical decompositions that avoid imported quantity data.

Finally, I contribute to work on sufficient statistics for general equilibrium (Costinot & Rodríguez-Clare, 2014; Caliendo & Parro, 2015; Adao et al., 2017; Baqaee & Farhi, 2019; Allen et al., 2020; Caliendo et al., 2022). This literature uses observed shares, elasticities, and network structure to discipline counterfactuals. I use the same equilibrium-in-changes discipline and the sufficient statistics but for cluster-level real value added.

The rest of the paper proceeds as follows. Section 2 defines the environment, value-added identities, and cluster centralities. Section 3 derives the incidence decomposition, the local-factor-market benchmark, and the comparison with Baqaee & Farhi (2024). Section 4 implements the decomposition on U.S. sectoral data, studying TFP contributions, comovement, and the GFC. Section 5 compares partial-equilibrium incidence statistics with full nested-CES general-equilibrium responses. Section 6 concludes.

2 General Framework

Following Rojas-Bernal (2026), this section studies a static, nonparametric general equilibrium economy with constant-returns-to-scale (CRS), comprising N firms and H household types. Let $\mathscr{H} = \{1, \dots, H\}$ index household types. Each tax authority

$g \in \mathscr{G}$ governs a subset $\mathscr{H}_g \subseteq \mathscr{H}$ and levies taxes and transfers within $\mathscr{H}_g$. Each firm $i \in \mathscr{N} = \{1, \cdots, N\}$ produces a good or service consumed by households or used as an intermediate by other firms. Let $\{\mathscr{N}_r\}_{r\in\mathscr{R}}$ be a partition of $\mathscr{N}$: $\mathscr{N}_r \cap \mathscr{N}_{r'} = \emptyset$ for $r \neq r'$ and $\bigcup_{r\in\mathscr{R}} \mathscr{N}_r = \mathscr{N}$. The goal is to characterize first-order changes in real value added for any firm cluster $\mathscr{N}_r \subseteq \mathscr{N}$, for any partition $\{\mathscr{N}_r\}_{r\in\mathscr{R}}$, using numeraire-invariant sufficient statistics. This section also establishes the central accounting identities and network-adjusted centralities that will later be used to characterize first-order changes in cluster-level real value added.

2.1 The Environment

Firms. Firm $i \in \mathscr{N}$ operates $y_i = A_i\, Q_i(L_i, X_i)$, where $L_i = Q_i^{\ell}(\{\ell_{ih}\}_{h\in\mathscr{H}})$ and $X_i = Q_i^{x}(\{x_{ij}\}_{j\in\mathscr{N}})$. A_i is a Hicks-neutral productivity; L_i and X_i are primary-factor and intermediate-input composites; ℓ_{ih} is factor h used by firm i; and x_{ij} is input j used by firm i. Given $\{p_j\}_{j\in\mathscr{N}}$ and $\{w_h\}_{h\in\mathscr{H}}$, firm i minimizes $\bar{c}_i = p_i^{\ell}\, L_i + p_i^{x}\, X_i = \sum_{h\in\mathscr{H}} w_h\, \ell_{ih} + \sum_{j\in\mathscr{N}} p_j\, x_{ij}$, where p_i^{ℓ} and p_i^{x} are the unit-cost indices dual to Q_i^{ℓ} and Q_i^{x}. Firms face an exogenous wedge: marginal cost satisfies $mc_i \equiv \partial\, \bar{c}_i / \partial\, y_i = \mu_i\, p_i$, with $\mu_i \in (0, 1]$. When $\mu_i < 1$, a share $1 - \mu_i$ of revenue $S_i = p_i\, y_i$ is rebated as profits π_i.

Households. Household h chooses $(C_h, L_h, \{C_{hi}\}_{i\in\mathscr{N}})$ to maximize $U_h(C_h, L_h)$, subject to $C_h = Q_h^c(\{C_{hi}\}_{i\in\mathscr{N}})$ and $E_h \equiv p_h^c C_h = \sum_{i\in\mathscr{N}} p_i C_{hi} \leq J_h + \Pi_h + T_h$, where C_{hi} is household h's consumption of good i, p_h^c is the expenditure index dual to Q_h^c, $J_h = w_h L_h$, $\Pi_h = \sum_{i\in\mathscr{N}} \kappa_{ih}\, \pi_i$, and T_h is a lump-sum transfer. Ownership shares satisfy $\kappa_{ih} \geq 0$ and $\sum_{h\in\mathscr{H}} \kappa_{ih} = 1$ for each i. Households differ in preferences, factor supplies, and equity holdings. Markets are incomplete: no state-contingent claims span idiosyncratic shocks to households or firms. Households cannot fully insure factor income or dividend risk, so exposure to shocks differs across types.

Market Clearing. Good and factor markets clear, and each fiscal authority satisfies budget neutrality. Goods market clearing: $y_i = \sum_{h\in\mathscr{H}} C_{hi} + \sum_{j\in\mathscr{N}} x_{ji}$ for all $i \in \mathscr{N}$. Factor markets clearing: $L_h = \sum_{i\in\mathscr{N}} \ell_{ih}$ for all $h \in \mathscr{H}$. Fiscal neutrality: $\sum_{h\in\mathscr{H}_g} T_h = 0$ for all $g \in \mathscr{G}$. Equilibrium is as in Rojas-Bernal (2026).

2.2 Value Added, Income, and External Resource Gaps

Nominal value added for cluster $\mathscr{N}_r$ is

$$GDP_r = \sum_{i\in\mathscr{N}_r} (1 - \mu_i\, \omega_i^x)\, S_i = \sum_{i\in\mathscr{N}_r} \left(\sum_{h\in\mathscr{H}} w_h\, \ell_{ih} + (1 - \mu_i)\, S_i \right). \tag{1}$$

Here $\omega_i^x \equiv \frac{\partial \log \bar{c}_i}{\partial \log p_i^x}$; by Shephard's lemma, $\omega_i^x = p_i^x X_i / \bar{c}_i$ is the intermediate input cost share. In equation (1), the first expression is the value-added representation: revenue net of intermediate input expenditure. The second is the income representation: primary-factor payments plus rents generated by cluster firms.

When cluster $\mathscr{N}_r$ has an associated household set $\mathscr{H}_r \subseteq \mathscr{H}$, define gross national income as $GNI_r \equiv \sum_{h\in\mathscr{H}_r}(J_h + \Pi_h)$ and net current transfers as $NCT_r \equiv \sum_{h\in\mathscr{H}_r} T_h$. Net primary income is $NPI_r \equiv GNI_r - GDP_r$, and gross national disposable income is $GNDI_r \equiv GNI_r + NCT_r$. The external resource gap is $GNDI_r - GDP_r$; it equals the net inflow from outside through factor payments, investment income, and transfers:

$$\overbrace{\sum_{\substack{h\in\mathscr{H}_r\\ i\notin\mathscr{N}_r}} w_h\,\ell_{ih} - \sum_{\substack{i\in\mathscr{N}_r\\ h\notin\mathscr{H}_r}} w_h\,\ell_{ih}}^{\text{Net factor income}_r} + \overbrace{\sum_{\substack{h\in\mathscr{H}_r\\ i\notin\mathscr{N}_r}} \kappa_{ih}\,(1-\mu_i)\,S_i - \sum_{i\in\mathscr{N}_r}(1-\sum_{h\in\mathscr{H}_r}\kappa_{ih})(1-\mu_i)S_i}^{\text{Net investment income}_r} + NCT_r.$$

For geographic partitions (country, region, states), $GNDI_r - GDP_r$ denotes the trade balance deficit under balance-of-payments accounting.

For cluster $\mathscr{N}_r$, the external resource gap is zero iff $NPI_r + NCT_r = 0$. When $NCT_r = 0$, $GNI_r = GDP_r$ (equivalently, $NPI_r = 0$) under four benchmark cases, each imposing *fully local factor markets*: (i) resident factors are not used outside the cluster, $\ell_{ih} = 0$ for $i \notin \mathscr{N}_r$ and $h \in \mathscr{H}_r$; and (ii) cluster firms do not hire nonresidents, $\ell_{ih} = 0$ for $i \in \mathscr{N}_r$ and $h \notin \mathscr{H}_r$. Conditional on these two restrictions, the four cases are: (i) *no rents* ($\mu_i = 1$ for $i \in \mathscr{N}$), e.g., first best; (ii) *home bias* ($\kappa_{ih} = 0$ for $h \in \mathscr{H}_r$ and $i \notin \mathscr{N}_r$) and *full local ownership* ($\sum_{h\in\mathscr{H}_r}\kappa_{ih} = 1$ for $i \in \mathscr{N}_r$), e.g., stringent capital controls; (iii) *home bias* ($\kappa_{ih} = 0$ for $h \in \mathscr{H}_r$ and $i \notin \mathscr{N}_r$) and *no domestic rents* ($\mu_i = 1$ for $i \in \mathscr{N}_r$), e.g., competitive local firms with restricted resident access to foreign capital markets; and (iv) *full local ownership* ($\sum_{h\in\mathscr{H}_r}\kappa_{ih} = 1$ for $i \in \mathscr{N}_r$) and *no external rents* ($\mu_i = 1$ for $i \notin \mathscr{N}_r$), e.g., competitive foreign firms with restricted foreign access to local capital markets. For the aggregate economy, primary income flows net to zero, so $GDP \equiv GNI$.

2.3 Cluster-Level Centralities

Tables 1, 2 and 3 summarize the measures introduced in this subsection. Let $\beta \equiv (\beta_{hi})_{h\in\mathscr{H},i\in\mathscr{N}}$ denote the household expenditure-share matrix. Define $\beta_{hi} \equiv \frac{\partial \log E_h}{\partial \log p_i}$; by Shephard's lemma, $\beta_{hi} = p_i C_{hi}/E_h$. Household h's direct expenditure intensity on cluster $\mathscr{N}_r$ is $\beta_{h|r} \equiv \sum_{i\in\mathscr{N}_r}\beta_{hi}$. Define revenue-based upstream centralities $\Omega_\ell \equiv (\Omega_{ih}^\ell)_{i\in\mathscr{N},h\in\mathscr{H}}$ and $\Omega_x \equiv (\Omega_{ij}^x)_{i\in\mathscr{N},j\in\mathscr{N}}$. Here $\Omega_{ih}^\ell \equiv \frac{\partial \log S_i}{\partial \log w_h}$ and $\Omega_{ij}^x \equiv \frac{\partial \log S_i}{\partial \log p_j}$; by

Shephard's lemma, $\Omega^{\ell}_{ih} = w_h\, \ell_{ih}/S_i$ and $\Omega^{x}_{ij} = p_j\, x_{ij}/S_i$. Firm i's direct intermediate input intensity on cluster $\mathcal{N}_r$ is $\Omega^{x}_{i|r} = \sum_{j\in\mathcal{N}_r} \Omega^{x}_{ij}$. The profit-share matrix Ω_π with element $\Omega^{\pi}_{ih} = \kappa_{ih}\, \pi_i/S_i$ captures the share of revenue from firm i reaching household h via rebated profits. Revenue adds up: $\sum_{h\in\mathcal{H}}(\Omega^{\ell}_{ih} + \Omega^{\pi}_{ih}) + \sum_{j\in\mathcal{N}} \Omega^{x}_{ij} = 1$. For any partition $\{\mathcal{N}_r\}_{r\in\mathcal{R}}$, $\sum_{r\in\mathcal{R}} \beta_{h|r} = 1$ and $\sum_{r\in\mathcal{R}} \Omega^{x}_{i|r} = \mu_i\, \omega^{x}_i$.

Intermediate-input and factor cost shares are collected in $\widetilde{\Omega}_x = (\widetilde{\Omega}^{x}_{ij})_{i,j\in\mathcal{N}}$ and $\widetilde{\Omega}_\ell = (\widetilde{\Omega}^{\ell}_{ih})_{i\in\mathcal{N},h\in\mathcal{H}}$, where $\widetilde{\Omega}^{x}_{ij} \equiv \frac{\partial \log \bar{c}_i}{\partial \log p_j} = p_j x_{ij}/\bar{c}_i$, and $\widetilde{\Omega}^{\ell}_{ih} \equiv \frac{\partial \log \bar{c}_i}{\partial \log w_h} = w_h \ell_{ih}/\bar{c}_i$. Let $\widetilde{\Psi}_x \equiv (I - \widetilde{\Omega}_x)^{-1}$ denote the downstream cost Leontief inverse, so that $\widetilde{\psi}^{x}_{ij}$ is firm i's total cost exposure to supplier j. Let $\widetilde{\Psi}_\ell \equiv \widetilde{\Psi}_x \widetilde{\Omega}_\ell$, so that $\widetilde{\psi}^{\ell}_{ih}$ is factor h's total cost share in firm i's production. Since all costs trace back to primary factors, $\sum_{h\in\mathcal{H}} \widetilde{\psi}^{\ell}_{ih} = 1$.

The rent-adjusted network exposures of firm i to supplier j and factor h are

$$\widetilde{\psi}^{x}_{ij,\Delta\Omega} = \sum_{m\in\mathcal{N}} (\widetilde{\Omega}^{x}_{im} - \Omega^{x}_{im})\widetilde{\psi}^{x}_{mj} = (1-\mu_i) \sum_{m\in\mathcal{N}} \widetilde{\Omega}^{x}_{im}\widetilde{\psi}^{x}_{mj},$$

$$\widetilde{\psi}^{\ell}_{ih,\Delta\Omega} = \sum_{m\in\mathcal{N}} (\widetilde{\Omega}^{x}_{im} - \Omega^{x}_{im})\widetilde{\psi}^{\ell}_{mh} = (1-\mu_i) \sum_{m\in\mathcal{N}} \widetilde{\Omega}^{x}_{im}\widetilde{\psi}^{\ell}_{mh}.$$

Both objects measure how shocks affect rents at firm i through its intermediate-input network. With perfect competition ($\mu_i = 1$), rent-adjusted exposure vanishes $\widetilde{\psi}^{x}_{ij,\Delta\Omega} = 0$ and $\widetilde{\psi}^{\ell}_{ih,\Delta\Omega} = 0$. With markups ($\mu_i < 1$), firm i prices above marginal cost and distributes a share $1 - \mu_i$ of revenue as rents. The objects $\widetilde{\psi}^{x}_{ij,\Delta\Omega}$ and $\widetilde{\psi}^{\ell}_{ih,\Delta\Omega}$ aggregate the paths through which marginal shocks to supplier j and factor h affect rents at firm i: firm i loads on each intermediate-input supplier m with weight $\widetilde{\Omega}^{x}_{im}$, and supplier m's downstream exposure is $\widetilde{\psi}^{x}_{mj}$ for shocks to supplier j and $\widetilde{\psi}^{\ell}_{mh}$ for shocks to factor h.

For this reason, define the firm-incidence and factor-incidence weights

$$\ddot{\lambda}^{r}_{j} = \mathbb{1}_{\{j\in\mathcal{N}_r\}} \frac{\lambda_j}{\Phi_r} + \sum_{i\in\mathcal{N}_r} \frac{\lambda_i}{\Phi_r}\, \widetilde{\psi}^{x}_{ij,\Delta\Omega},$$

$$\ddot{\Lambda}^{r}_{h} = \sum_{i\in\mathcal{N}_r} \frac{\lambda_i}{\Phi_r} \left(\widetilde{\Omega}^{\ell}_{ih} + \widetilde{\psi}^{\ell}_{ih,\Delta\Omega}\right).$$

The first statistic measures the rent-adjusted share of cluster-r value added that passes through firm j; the second measures the rent-adjusted share of value added traced to factor h. Each statistic has a direct component and an indirect rent-scaled component. The direct term captures value added generated inside the cluster: firm j's own production in $\ddot{\lambda}^{r}_{j}$ and factor h's direct cost share in $\ddot{\Lambda}^{r}_{h}$. The indirect term captures rents generated by cluster firms whose production uses inputs sourced, directly or indirectly, from firm j or from supply chains intensive in factor h. Under perfect competition within the cluster ($\mu_i = 1$ for all $i \in \mathscr{N}_r$), these indirect terms vanish. Both statistics weight the rent-adjusted network exposure by λ_i/Φ_r. Moreover, $\{\ddot{\Lambda}^r_h\}_{h\in\mathscr{H}}$ is a probability distribution over factors for cluster r, since $\sum_{h\in\mathscr{H}} \ddot{\Lambda}^r_h = 1$.

Table 1: Direct Centralities

	Definition	*In equilibrium*	*Property*
β	$\beta_{hi} \equiv \frac{\partial \log E_h}{\partial \log p_i}$	Expenditure share on i	$\sum_{i\in\mathscr{N}} \beta_{hi} = 1$
κ	$\kappa_{ih} = \frac{\partial \Pi_h}{\partial \pi_i}$	Ownership share of h in i	$\sum_{h\in\mathscr{H}} \kappa_{ih} = 1$
$\widetilde{\Omega}_\ell$	$\widetilde{\Omega}^\ell_{ih} \equiv \frac{\partial \log \bar{c}_i}{\partial \log w_h}$	Cost share of ℓ_{ih}	$\sum\limits_{h\in\mathscr{H}} \widetilde{\Omega}^\ell_{ih} + \sum\limits_{j\in\mathscr{N}} \widetilde{\Omega}^x_{ij} = 1$
$\widetilde{\Omega}_x$	$\widetilde{\Omega}^x_{ij} \equiv \frac{\partial \log \bar{c}_i}{\partial \log p_j}$	Cost share of x_{ij}	
ω_x	$\omega^x_i \equiv \frac{\partial \log \bar{c}_i}{\partial \log p^x_i}$	Cost share of X_i	$\omega^x_i = \sum_{j\in\mathscr{N}} \widetilde{\Omega}^x_{ij}$
Ω_ℓ	$\Omega^\ell_{ih} \equiv \frac{\partial \log S_i}{\partial \log w_h}$	Share of S_i spent on ℓ_{ih}	$\Omega^\ell_{ih} = \mu_i \widetilde{\Omega}^\ell_{ih}$
Ω_x	$\Omega^x_{ij} \equiv \frac{\partial \log S_i}{\partial \log p_j}$	Share of S_i spent on x_{ij}	$\Omega^x_{ij} = \mu_i \widetilde{\Omega}^x_{ij}$
Ω_π	$\Omega^\pi_{ih} \equiv \kappa_{ih}\frac{\pi_i}{S_i}$	Share of S_i rebated as $\kappa_{ih}\pi_i$	$\sum\limits_{h\in\mathscr{H}} (\Omega^\ell_{ih} + \Omega^\pi_{ih}) + \sum\limits_{j\in\mathscr{N}} \Omega^x_{ij} = 1$

Table 2: Network-Adjusted Centralities

Matrix	*Equilibrium Expression*	*Properties*
Downstream / Cost-Based Centralities		
$\widetilde{\Psi}_x = (I - \widetilde{\Omega}_x)^{-1}$	$\widetilde{\psi}^x_{ij}$: Cost exposure of firm i to firm j	
$\widetilde{\Psi}_\ell = \widetilde{\Psi}_x \widetilde{\Omega}_\ell$	$\widetilde{\psi}^\ell_{ih}$: factor h's value-added share in $\bar{c}_i$	$\sum_{h\in\mathscr{H}} \widetilde{\psi}^\ell_{ih} = 1$
Upstream / Revenue-Based Centralities		
$\Psi_x = (I - \Omega_x)^{-1}$	ψ^x_{ij}: Share of S_i accruing to firm j	
$\mathscr{B} = \beta\Psi_x$	$\mathscr{B}_{hi}$: Share of E_h accruing to firm i	
$\Psi_\ell = \Psi_x \Omega_\ell$	ψ^ℓ_{ih}: Share of S_i accruing to factor h	
$\mathscr{C} = \beta\Psi_\ell$	$\mathscr{C}_{hb}$: Share of E_h accruing to factor b	
$\lambda = \mathscr{B}'\chi$	λ_i: Aggregate sales share S_i/GDP	$\sum_{i\in\mathscr{N}} \lambda_i \geq 1$
$\Lambda = \mathscr{C}'\chi$	Λ_h: Factor income share J_h/GDP	$\sum_{h\in\mathscr{H}} \Lambda_h \leq 1$
$\chi = (\Omega_\ell + \Omega_\pi)'\lambda + \tau$	χ_h: Expenditure share E_h/GDP	$\sum_{h\in\mathscr{H}} \chi_h = 1$

Define upstream (revenue-based) matrices $\Psi_x \equiv (I - \Omega_x)^{-1}$ (firm-to-firm), $\Psi_\ell \equiv \Psi_x\Omega_\ell$ (firm-to-factor), $\mathscr{B} \equiv \beta\Psi_x$ (household-to-firm), and $\mathscr{C} \equiv \beta\Psi_\ell$ (household-to-factor). Their elements ψ^x_{ij}, ψ^ℓ_{ih}, $\mathscr{B}_{hi}$, and $\mathscr{C}_{hb}$ are shares of (i) firm i's revenue reaching firm j, (ii) firm i's revenue reaching factor h, (iii) household h's expenditure reaching firm i, and (iv) household h's expenditure reaching factor b. In equilibrium, Domar weights λ, factor-income shares $\Lambda_h \equiv J_h/GDP$, and expenditure shares $\chi_h \equiv E_h/GDP$ satisfy $\lambda = \mathscr{B}'\chi$, $\Lambda = \mathscr{C}'\chi$, and $\chi_h = \Lambda_h + \sum_{i\in\mathscr{N}} \kappa_{ih}(1-\mu_i)\lambda_i + \tau_h$, where $\tau_h \equiv$

Table 3: Centralities for Cluster r

Concept	*Definition*	*Properties*
Nominal value added	$GDP_r = \sum_{i\in\mathcal{N}_r}(1-\mu_i\,\omega^x_i)S_i$	$GDP = \sum_{r\in\mathcal{R}} GDP_r$
Value added share	$\Phi_r = GDP_r/GDP$	$\sum_{r\in\mathcal{R}}\Phi_r = 1$
Expenditure intensity	$\beta_{h\|r} = \sum_{i\in\mathcal{N}_r}\beta_{hi}$	$\sum_{r\in\mathcal{R}}\beta_{h\|r} = 1$
Intermediate intensity	$\Omega^x_{i\|r} = \sum_{j\in\mathcal{N}_r}\Omega^x_{ij}$	$\sum_{r\in\mathcal{R}}\Omega^x_{i\|r} = \mu_i\,\omega^x_i$
Network exposure of firm i to firm j	$\widetilde{\psi}^x_{ij,\Delta\Omega} = \sum_{m\in\mathcal{N}}(\widetilde{\Omega}^x_{im}-\Omega^x_{im})\,\widetilde{\psi}^x_{mj}$	
Share of value added passing through firm j	$\ddot{\lambda}^r_j = \mathbb{1}_{\{j\in\mathcal{N}_r\}}\frac{\lambda_j}{\Phi_r} + \sum_{i\in\mathcal{N}_r}\frac{\lambda_i}{\Phi_r}\,\widetilde{\psi}^x_{ij,\Delta\Omega}$	
Network exposure of firm i to factor h	$\widetilde{\psi}^\ell_{ih,\Delta\Omega} = \sum_{j\in\mathcal{N}}(\widetilde{\Omega}^x_{ij}-\Omega^x_{ij})\,\widetilde{\psi}^\ell_{jh}$	
Share of value added attributed to h	$\ddot{\Lambda}^r_h = \sum_{i\in\mathcal{N}_r}\frac{\lambda_i}{\Phi_r}(\widetilde{\Omega}^\ell_{ih}+\widetilde{\psi}^\ell_{ih,\Delta\Omega})$	$\sum_{h\in\mathcal{H}}\ddot{\Lambda}^r_h = 1$
Distortion centrality for h	$\delta^r_h = \ddot{\Lambda}^r_h/\Lambda_h$	
Expenditure centrality for h	$M^r_h = \sum_{b\in\mathcal{H}}\mathscr{C}_{hb}\,\delta^r_b$	
Revenue centrality for i	$F^r_i = \sum_{h\in\mathcal{H}}\psi^\ell_{ih}\,\delta^r_h$	

T_h/GDP. This yields a $2H+N$ system; impose $\sum_{h\in\mathcal{H}}\chi_h = 1$ by Walras' law.

For factor h, define $\delta^r_h \equiv \ddot{\Lambda}^r_h/\Lambda_h$. I refer to δ^r_h as *distortion centrality* with respect to r: it measures how large factor h's share of cluster-r value added is relative to its economywide income share—i.e., how undervalued h is in aggregate factor-income terms from cluster r's perspective. Define $M^r_h \equiv \sum_{b\in\mathcal{H}}\mathscr{C}_{hb}\,\delta^r_b$ and $F^r_i \equiv \sum_{h\in\mathcal{H}}\psi^\ell_{ih}\,\delta^r_h$. M^r_h is high when household h's expenditure reaches factors with high δ^r (high $\mathscr{C}_{hb}$ weight), and F^r_i is high when firm i's revenue ultimately accrues to factors with high δ^r (high ψ^ℓ_{ih} weight). Accordingly, I refer to M^r_h as household h's *expenditure centrality* with respect to r and to F^r_i as firm i's *revenue centrality* with respect to r.

3 Value-Added Incidence in Production Networks

This section derives a nonparametric decomposition of cluster-level real value added. The results show how productivity, markup, factor-supply, and demand-composition shocks affect real value added through network-adjusted sufficient statistics. I proceed in four steps: first, I derive cluster-level price pass-through; second, I decompose real value added; third, I compare the resulting formulas with open-economy real-GDP decompositions; and fourth, I illustrate their implementation in two examples.

3.1 Prices

Proposition 1 characterizes downstream pass-through of productivity, inverse-markup, and factor-cost shocks to firm prices and the cluster-r value-added deflator, with elasticities given by cost-based centralities.

Proposition 1. The first-order responses of firm prices and the cluster-r value-added deflator to productivity, inverse-markup, and factor-cost shocks are

$$d\log p_i = -\sum_{j\in\mathcal{N}} \widetilde{\psi}^x_{ij}\, d\log(A_j\,\mu_j) + \sum_{h\in\mathcal{H}} \widetilde{\psi}^\ell_{ih}\, d\log w_h,$$

$$d\log p_{Y_r} = -\sum_{i\in\mathcal{N}} \ddot{\lambda}^r_i\, d\log(A_i\,\mu_i) + \sum_{h\in\mathcal{H}} \ddot{\Lambda}^r_h\, d\log w_h.$$

The proposition has six implications. First, firm prices respond directly to productivity, inverse-markup, and factor-cost shocks; demand reallocation affects prices only indirectly through factor prices. Second, firm-level elasticities are governed by cost-based centralities, $\widetilde{\psi}^x_{ij}$ and $\widetilde{\psi}^\ell_{ih}$. Third, the cluster-$r$ deflator aggregates firm-level pass-through using the cluster-specific incidence weights $\ddot{\lambda}^r_i$ and $\ddot{\Lambda}^r_h$. Fourth, distortions generate cross-cluster price spillovers. If firms in cluster r generate no rents ($\mu_i = 1$ for all $i \in \mathcal{N}_r$), then $\ddot{\lambda}^r_j = \mathbb{1}_{\{j\in\mathcal{N}_r\}}\lambda_j/\Phi_r$, so a firm-level productivity shock affects p_{Y_r} directly only when $j \in \mathcal{N}_r$. With markups, indirect terms transmit shocks across clusters through rent-adjusted downstream exposure. Fifth, a common increase in A or μ lowers p_{Y_r} in proportion to the cluster-specific network multiplier $\sum_{i\in\mathcal{N}} \ddot{\lambda}^r_i$, while a common increase in factor prices raises p_{Y_r} one-for-one because $\sum_{h\in\mathcal{H}} \ddot{\Lambda}^r_h = 1$. Sixth, aggregating across clusters with weights Φ_r recovers the economy-wide pass-through: $d\log p_Y = -\widetilde{\lambda}'\, d\log(A\mu) + \widetilde{\Lambda}'\, d\log w$.

3.2 Value-Added Incidence Decomposition

Theorem 1 is the main result. It shows that cluster real-value-added growth can be decomposed into technology incidence, factor-supply incidence, competitiveness, and distributional reallocation.

Theorem 1. Cluster-level real value added with heterogeneous households.

(a) In equilibrium, cluster-level real value added satisfies

$$Y_r \;=\; TFP_r\; F_r(\{L_h\}_{h\in\mathcal{H}})\,, \tag{2}$$

where TFP_r denotes total factor productivity; F_r satisfies $\partial\log F_r/\partial\log L_h = \ddot{\Lambda}^r_h$.

(b) Around equilibrium

$$d \log Y_r \;=\; d \log TFP_r \;+\; \sum_{h \in \mathcal{H}} \ddot{\Lambda}^r_h \, d \log L_h, \tag{3}$$

$$d \log TFP_r \;=\; \mathit{Technology}_r \;+\; \mathit{Allocative\ Efficiency}_r, \tag{4}$$

$$\mathit{Technology}_r = \sum_{i \in \mathcal{N}} \ddot{\lambda}^r_i \; d \, \log \, A_i, \qquad \begin{array}{c}\mathit{Allocative} \\ \mathit{Efficiency}_r\end{array} = \overbrace{\sum_{i \in \mathcal{N}} \ddot{\lambda}^r_i \; d \, \log \, \mu_i}^{\mathit{Competitiveness}_r} + \begin{array}{c}\mathit{Distributional} \\ \mathit{Reallocation}_r\end{array}.$$

(c) *Distributional Reallocation*$_r$ has the following equivalent representations:

$$1. \quad d \log \Phi_r - \overbrace{\sum_{h \in \mathcal{H}} \delta^r_h \; d \, \Lambda_h}^{\mathit{Entropic}\ TT_r}, \qquad 2. \quad d \log \Phi_r - \overbrace{\sum_{h \in \mathcal{H}} M^r_h \; d \, \chi_h}^{\mathit{Distributive}\ TT_r} - \overbrace{\sum_{h \in \mathcal{H}} \chi_h \sum_{b \in \mathcal{H}} \delta^r_b \; d \, \mathscr{C}_{hb}}^{\mathit{Centrality}\ TT_r},$$

$$3. \quad d \log \Phi_r - \sum_{h \in \mathcal{H}} M^r_h \; d \, \chi_h - \overbrace{\sum_{h \in \mathcal{H}} \chi_h \sum_{i \in \mathcal{N}} F^r_i \; d \, \beta_{hi}}^{\mathit{Final\ Demand}\ TT_r} - \overbrace{\sum_{i \in \mathcal{N}} \mu_i \, \lambda_i \sum_{j \in \mathcal{N}} F^r_j \; d \, \widetilde{\Omega}^x_{ij}}^{\mathit{Intermediate\ Demand}\ TT_r}$$

$$- \overbrace{\sum_{i \in \mathcal{N}} \mu_i \, \lambda_i \sum_{h \in \mathcal{H}} \delta^r_h \; d \, \widetilde{\Omega}^\ell_{ih}}^{\mathit{Factor\ Demand}\ TT_r} - \overbrace{\sum_{i \in \mathcal{N}} \lambda_i \; F^r_i \; d \, \log \, \mu_i}^{\mathit{Competitive}\ TT_r}.$$

(d) The variations for the value-added share Φ_r and expenditure share χ_h are

$$\begin{aligned} d \, \Phi_r = & \sum_{h \in \mathcal{H}} \left(\beta_{h|r} + \mathscr{B}_{h|r} \right) d \, \chi_h + \sum_{i \in \mathcal{N}} \lambda_i \, \psi_{i|r} \; d \log \mu_i + \sum_{i \in \mathcal{N}} \mu_i \, \lambda_i \sum_{j \in \mathcal{N}} \psi_{j|r} \; d \, \widetilde{\Omega}^x_{ij} \\ & + \sum_{h \in \mathcal{H}} \chi_h \left(d \, \beta_{h|r} + \sum_{i \in \mathcal{N}} \psi_{i|r} \; d \, \beta_{hi} \right) + \sum_{i \notin \mathcal{N}_r} \left(\lambda_i \; d \, \Omega^x_{i|r} - \sum_{j \in \mathcal{N}_r} \lambda_j \; d \, \Omega^x_{ji} \right), \end{aligned} \tag{5}$$

$$d \, \chi_h = d \, \Lambda_h + \sum_{i \in \mathcal{N}} \left((1 - \mu_i) \left(\lambda_i \; d \, \kappa_{ih} + \kappa_{ih} \; d \, \lambda_i \right) - \kappa_{ih} \; \lambda_i \; d \, \mu_i \right) + d \, \tau_h, \tag{6}$$

where

$$\mathscr{B}_{h|r} = \sum_{i \notin \mathcal{N}_r} \mathscr{B}_{hi} \, \Omega^x_{i|r} - \sum_{i \in \mathcal{N}_r} \mathscr{B}_{hi} \sum_{j \notin \mathcal{N}_r} \Omega^x_{ij}, \quad \psi_{i|r} = \sum_{m \notin \mathcal{N}_r} \psi^x_{im} \; \Omega^x_{m|r} - \sum_{m \in \mathcal{N}_r} \psi^x_{im} \sum_{f \notin \mathcal{N}_r} \Omega^x_{mf}.$$

From equation (2), equilibrium cluster-level real value added Y_r equals TFP_r times a CRS aggregator F_r over primary factors, with elasticities given by the cluster value-added distribution $\ddot{\Lambda}^r = \{\ddot{\Lambda}^h_1\}_{h \in \mathcal{H}}$. Equation (3) decomposes $d \log Y_r$ into a TFP_r component and a factor component. Equation (4) decomposes $d \log TFP_r$ into technology, competitiveness, and distributional reallocation. *Technology*$_r$ captures pure productivity effects, holding allocations fixed. *Competitiveness*$_r$ captures efficiency gains from wedge-driven reallocation; absent distributional changes, shocks to A_i or μ_i affect TFP_r in proportion to $\ddot{\lambda}^r_i$. *Distributional reallocation*$_r$ captures changes in TFP_r driven by endogenous shifts in the factor-income distribution that reallocate inputs across firms. Together, *competitiveness*$_r$ and *distributional reallocation*$_r$ comprise *allocative efficiency*$_r$.

Finally, aggregation is consistent: weighting equation (3) by Φ_r and adding across clusters recovers the economy wide decomposition $d\log Y = d\log TFP + \widetilde{\Lambda}'\, d\log L$.

Theorem 1 also provides three equivalent representations of *distributional reallocation*$_r$, offering complementary lenses on how the income distribution shapes TFP_r. All three emphasize that TFP_r rises with the value-added share Φ_r and with reallocations that ease cluster-specific bottlenecks. In the first definition, the *entropic terms of trade*$_r$ measures how changes in the distance between the cluster factor-incidence distribution $\ddot{\Lambda}^r$ and the economywide factor-income distribution Λ affect the efficiency with which factors generate cluster value added.[1] This mechanism operates through the cluster-specific distortion centralities $\delta^r_h \equiv \ddot{\Lambda}^r_h/\Lambda_h$. A high value of δ^r_h means that factor h is more important for producing Y_r than its economywide income share suggests. Hence, holding $\ddot{\Lambda}^r$ fixed, TFP_r rises when factor income shifts away from high-δ^r factors toward low-δ^r factors. For example, if $\delta^r_h > \delta^r_b$, then a shift in factor income from h to b raises TFP_r. Intuitively, high-δ^r factors are cluster-specific bottlenecks. Reducing their relative income share lowers their incidence-adjusted cost and allows resources to reallocate toward the supply chains that generate cluster-r value added.

The second representation splits the *entropic terms of trade*$_r$ into two components: (i) expenditure-share shifts, $d\,\chi$, and (ii) consumer-to-factor centrality shifts, $d\,\mathscr{C}$. The *distributive terms of trade*$_r$ imply that cluster-r efficiency falls when expenditure shifts toward households with high expenditure centrality M^r_h. This statistic is $M^r_h = \mathscr{C}'_{\uparrow h}\,\delta^r$ where $\mathscr{C}_{\uparrow h} \equiv (\mathscr{C}_{h1}, \ldots, \mathscr{C}_{hH})'$ collects household h's consumer-to-factor centralities and $\delta^r \equiv (\delta^r_1, \ldots, \delta^r_H)'$ collects cluster-r's centralities. Consumer-to-factor centralities are high when a household h's consumption basket reaches a factor directly or indirectly through the production network. Hence, high M^r_h means that household h's basket is intensive in high-δ^r factors, which are bottlenecks in the production of Y_r. The vector $M^r \equiv (M^r_1, \ldots, M^r_H)'$ ranks households by the marginal drag on TFP_r from reallocating one percentage point of expenditure share toward them. Thus, M^r is the sufficient statistic for expenditure recomposition: TFP_r rises when expenditure shifts from household h to household b with $M^r_h > M^r_b$. The *centrality terms of trade*$_r$ capture changes in $\mathscr{C}$ holding expenditure shares fixed. An increase $d\mathscr{C}_{hb} > 0$ lowers TFP_r most when household h has a large expenditure share χ_h and factor b has high distortion

[1] Treat $\ddot{\Lambda}^r$ as a probability vector. The Kullback-Leibler divergence from $\ddot{\Lambda}^r$ to Λ is $\mathcal{KL}(\ddot{\Lambda}^r \mid \Lambda) = \sum_{h\in\mathscr{H}} \ddot{\Lambda}^r_h \log(\ddot{\Lambda}^r_h/\Lambda_h)$. Holding $\ddot{\Lambda}^r$ fixed, $d\,\mathcal{KL}(\ddot{\Lambda}^r \mid \Lambda) = -\sum_{h\in\mathscr{H}} \ddot{\Lambda}^r_h\, d\log\Lambda_h = -\sum_{h\in\mathscr{H}} \delta^r_h\, d\Lambda_h$.

centrality δ^r_b. This term isolates how changes in network transmission alter cluster efficiency through the factor-income distribution.

The final representation decomposes the *centrality terms of trade*$_r$ into four channels: final demand, intermediate demand, factor demand, and competitiveness. The *final*- and *intermediate*-demand terms of trade imply that TFP_r falls when spending shifts toward firms with high revenue centrality F^r_i. This statistic is $F^r_i = \psi^\ell_{i\cdot}\delta^r$, where $\psi^\ell_{i\cdot} \equiv (\psi^\ell_{i1}, \ldots, \psi^\ell_{iH})$ collects the shares of firm i's revenue that accrue, directly and indirectly, to each factor, and $\delta^r \equiv (\delta^r_1, \ldots, \delta^r_H)'$ collects cluster-r distortion centralities. A high F^r_i means that firm i's revenue ultimately reaches high-δ^r factors, which are bottlenecks in the production of Y_r. The vector $F^r \equiv (F^r_1, \ldots, F^r_N)'$ ranks firms by the marginal drag on TFP_r from reallocating demand toward them. Thus, F^r is the sufficient statistic for demand recomposition: TFP_r rises when households or firms shift spending from firm i to firm j with $F^r_i > F^r_j$. The *factor-demand* terms of trade capture substitution across factors with different cluster-r distortion centralities. The vector δ^r is the sufficient statistic for factor-cost recomposition: TFP_r rises when firm i shifts spending from factor h to factor b with $\delta^r_h > \delta^r_b$. More generally, replacing factor h with intermediate input j raises TFP_r when $\delta^r_h > F^r_j$. Finally, the *competitive* terms of trade capture the distributional effect of markup changes. Holding other margins fixed, an increase in μ_i lowers the distributional component of $\log TFP_r$ in proportion to $\lambda_i F^r_i$.

Equation (5) characterizes first-order variation in cluster r's value-added share, Φ_r. It identifies four margins. First, Φ_r rises when expenditure shifts toward households with high direct exposure to cluster output, $\beta_{h|r}$, or high indirect net exposure, $\mathscr{B}_{h|r}$. The latter measures household h's expenditure that reaches cluster-r revenue through external firms' demand for cluster inputs, net of expenditure that reaches external-firm revenue through cluster firms' input demand. Thus, a shift in expenditure from household b to household h raises Φ_r if $\beta_{h|r} + \mathscr{B}_{h|r} > \beta_{b|r} + \mathscr{B}_{b|r}$. Second, Φ_r responds to markup shocks through the cluster's revenue exposure to the affected firm. A marginal increase in μ_i changes Φ_r in proportion to $\lambda_i \psi_{i|r}$, where $\psi_{i|r}$ is firm i's net revenue exposure to cluster r: revenue reaching cluster firms through external intermediate demand, net of revenue reaching external firms through cluster-firm intermediate demand. Third, Φ_r responds to cost recomposition. Changes in firm i's intermediate-input cost share on supplier j, $\widetilde{\Omega}^x_{ij}$, affect Φ_r with weight $\mu_i \lambda_i \psi_{j|r}$; changes in direct cluster demand,

$\beta_{h|r}$, enter with weight χ_h; and changes in household h's demand for firm i, β_{hi}, enter with weight $\chi_h \psi_{i|r}$. Finally, Φ_r rises when the net routing of aggregate revenue toward cluster firms increases: external firms spend more on intermediates produced in $\mathscr{N}_r$, or cluster firms spend less on intermediates produced outside $\mathscr{N}_r$. Equation (6) gives the first-order variation in household expenditure shares, χ_h. It shows that $d\chi_h$ loads on factor income, profit income, ownership shares, firm size, and transfers. Profit income from firm i to household h scales with κ_{ih} and λ_i. Since μ_i is an inverse markup, a fall in μ_i raises rents and increases the expenditure share of households that own firm i.

Corollary 1. Partial equilibrium allocative efficiency effects for cluster $\mathscr{N}_r$.

(a) **Markups.** For a 1% increase in μ_i, equivalently a markup reduction:

$$\ddot{\lambda}_i^r + \frac{\lambda_i}{\Phi_r}\left(\psi_{i|r} + \mu_i\left(\mathbb{1}_{\{i\notin\mathscr{N}_r\}}\sum_{j\in\mathscr{N}_r}\widetilde{\Omega}_{ij}^x - \mathbb{1}_{\{i\in\mathscr{N}_r\}}\sum_{j\notin\mathscr{N}_r}\widetilde{\Omega}_{ij}^x\right)\right) - \lambda_i F_i^r.$$

(b) **Factor-income shares.** For a one-percentage-point shift from Λ_b to Λ_h:

$$\frac{1}{\Phi_r}\left(\beta_{h|r} - \beta_{b|r} + \mathscr{B}_{h|r} - \mathscr{B}_{b|r}\right) + \delta_b^r - \delta_h^r.$$

(c) **Expenditure shares.** For a one-percentage-point shift from χ_b to χ_h:

$$\frac{1}{\Phi_r}\left(\beta_{h|r} - \beta_{b|r} + \mathscr{B}_{h|r} - \mathscr{B}_{b|r}\right) + M_b^r - M_h^r.$$

(d) **Factoral cost reallocation.** For a one-percentage-point shift from $\widetilde{\Omega}_{ib}^\ell$ to $\widetilde{\Omega}_{ih}^\ell$:

$$\mu_i \lambda_i \left(\delta_b^r - \delta_h^r\right).$$

(e) **Final expenditure reallocation.** For a one-percentage-point shift from β_{hj} to β_{hi}:

$$\chi_h\left(F_j^r - F_i^r + \frac{1}{\Phi_r}\left(\psi_{i|r} - \psi_{j|r} + \mathbb{1}_{\{i\in\mathscr{N}_r\}} - \mathbb{1}_{\{j\in\mathscr{N}_r\}}\right)\right).$$

(f) **Intermediate cost reallocation.** For a one-percentage-point shift from $\widetilde{\Omega}_{ij}^x$ to $\widetilde{\Omega}_{im}^x$:

$$\mu_i \lambda_i\left(F_j^r - F_m^r + \frac{1}{\Phi_r}\left(\psi_{m|r} - \psi_{j|r} + \mathbb{1}_{\{m\in\mathscr{N}_r\}} - \mathbb{1}_{\{j\in\mathscr{N}_r\}}\right)\right).$$

Corollary 1 gives six partial-equilibrium comparative statics for the allocative-efficiency component of TFP_r, holding all other channels fixed. First, a reduction in firm i's markup raises TFP_r if $\frac{\ddot{\lambda}_i^r}{\lambda_i} + \frac{1}{\Phi_r}\left(\psi_{i|r} + \mu_i\left(\mathbb{1}_{\{i\notin\mathscr{N}_r\}}\sum_{j\in\mathscr{N}_r}\widetilde{\Omega}_{ij}^x - \mathbb{1}_{\{i\in\mathscr{N}_r\}}\sum_{j\notin\mathscr{N}_r}\widetilde{\Omega}_{ij}^x\right)\right) > F_i^r$. A large F_i^r makes this condition harder to satisfy because the competitive terms-of-trade effect is a larger distributional drag. Second, a one-percentage-point shift in factor income from factor b to factor h raises TFP_r if $\beta_{h|r} - \beta_{b|r} + \mathscr{B}_{h|r} - \mathscr{B}_{b|r} > \Phi_r(\delta_h^r - \delta_b^r)$.

Third, a one-percentage-point shift in expenditure share from household b to household h raises TFP_r if $\beta_{h|r} - \beta_{b|r} + \mathscr{B}_{h|r} - \mathscr{B}_{b|r} > \Phi_r(M_h^r - M_b^r)$. Thus, factor-income and expenditure reallocation are not equivalent: for the same shift from b to h, the expenditure-share effect exceeds the factor-income effect when $M_b^r - M_h^r > \delta_b^r - \delta_h^r$. Fourth, a one-percentage-point shift in firm i's factor-cost share from factor b to factor h, $\widetilde{\Omega}_{ib}^{\ell} \to \widetilde{\Omega}_{ih}^{\ell}$, raises TFP_r if $\delta_b^r > \delta_h^r$. Fifth, a one-percentage-point shift in household h's final expenditure share from firm j to firm i, $\beta_{hj} \to \beta_{hi}$, raises TFP_r if $\psi_{i|r} - \psi_{j|r} + \mathbb{1}_{\{i \in \mathscr{N}_r\}} - \mathbb{1}_{\{j \in \mathscr{N}_r\}} > \Phi_r(F_i^r - F_j^r)$. Sixth, a one-percentage-point shift in firm i's intermediate-input cost share from supplier j to supplier m, $\widetilde{\Omega}_{ij}^{x} \to \widetilde{\Omega}_{im}^{x}$, raises TFP_r if $\psi_{m|r} - \psi_{j|r} + \mathbb{1}_{\{m \in \mathscr{N}_r\}} - \mathbb{1}_{\{j \in \mathscr{N}_r\}} > \Phi_r(F_m^r - F_j^r)$. In each case, TFP_r rises when the value-added-share gain from reallocation exceeds the allocative-efficiency drag from shifting weight toward high-δ^r, high-M^r, or high-F^r objects.

When cluster r is associated with a set of households $\mathscr{H}_r$ and *fully local factor markets* hold—resident factors are not used outside the cluster, and cluster firms do not hire nonresidents—equation (1) implies $\Phi_r = \sum_{h \in \mathscr{H}_r} \Lambda_h + \sum_{i \in \mathscr{N}_r} (1-\mu_i)\lambda_i$ and $d\Phi_r = \sum_{h \in \mathscr{H}_r} d\Lambda_h + \sum_{i \in \mathscr{N}_r}((1-\mu_i)d\lambda_i - \lambda_i d\mu_i)$. Corollary 2 characterizes $d \log Y_r$ in this case.

Corollary 2. Suppose cluster r has *fully local factor markets*. Then

$$
\begin{aligned}
d \log Y_r = & \sum_{i \in \mathscr{N}} \ddot{\lambda}_i^r \, d \log A_i + \sum_{i \in \mathscr{N}_r} (1-\mu_i) \frac{\lambda_i}{\Phi_r} \Big(d \log(\mu_i \lambda_i) + \sum_{m \in \mathscr{N}} \widetilde{\Omega}_{im}^{x} \sum_{j \in \mathscr{N}} \widetilde{\psi}_{mj}^{x} \, d \log \mu_j \Big) \\
& - \sum_{i \in \mathscr{N}_r} (1-\mu_i) \frac{\lambda_i}{\Phi_r} \Big(\sum_{h \in \mathscr{H}_r} \widetilde{\Omega}_{ih}^{\ell} \, d \log \Lambda_h + \sum_{h \in \mathscr{H}} \sum_{j \in \mathscr{N}} \widetilde{\Omega}_{ij}^{x} \, \widetilde{\psi}_{jh}^{\ell} \, d \log \Lambda_h \Big) + \sum_{h \in \mathscr{H}} \ddot{\Lambda}_h^r \, d \log L_h.
\end{aligned} \tag{7}
$$

Equivalently, $d \log Y_r$ can be written as

$$
\begin{aligned}
d \log Y_r = & \sum_{i \in \mathscr{N}} \ddot{\lambda}_i^r \, d \log A_i + \sum_{i \in \mathscr{N}_r} (1-\mu_i) \frac{\lambda_i}{\Phi_r} d \log \mu_i + \sum_{h \in \mathscr{H}} \frac{\chi_h}{\Phi_r} \Big(\Xi_{\chi_h}^r \, d \log \chi_h + \sum_{j \in \mathscr{N}} \Xi_{\beta_{hj}}^r d\beta_{hj} \Big) \\
& + \sum_{j \in \mathscr{N}} \frac{\lambda_j}{\Phi_r} \Big(\Xi_{\mu_j}^r \, d \log \mu_j + \mu_j \Big(\sum_{m \in \mathscr{N}} \Xi_{\widetilde{\Omega}_{jm}^{x}}^r \, d\widetilde{\Omega}_{jm}^{x} - \sum_{h \in \mathscr{H}} \Xi_{\widetilde{\Omega}_{jh}^{\ell}}^r \, d\widetilde{\Omega}_{jh}^{\ell} \Big) \Big) + \sum_{h \in \mathscr{H}} \ddot{\Lambda}_h^r \, d \log L_h,
\end{aligned} \tag{8}
$$

$$
\Xi_{\chi_h}^r = \sum_{i \in \mathscr{N}_r} (1-\mu_i) \Big(\mathscr{B}_{hi} - \lambda_i \sum_{b \in \mathscr{H}} \frac{\mathscr{C}_{hb}}{\Lambda_b} \Big(\mathbb{1}_{\{b \in \mathscr{H}_r\}} \widetilde{\Omega}_{ib}^{\ell} + \sum_{j \in \mathscr{N}} \widetilde{\Omega}_{ij}^{x} \, \widetilde{\psi}_{jb}^{\ell} \Big) \Big),
$$

$$
\Xi_{\beta_{hj}}^r = \sum_{i \in \mathscr{N}_r} (1-\mu_i) \Big(\psi_{ji}^{x} - \lambda_i \sum_{b \in \mathscr{H}} \frac{\psi_{jb}^{\ell}}{\Lambda_b} \Big(\mathbb{1}_{\{b \in \mathscr{H}_r\}} \widetilde{\Omega}_{ib}^{\ell} + \sum_{m \in \mathscr{N}} \widetilde{\Omega}_{im}^{x} \, \widetilde{\psi}_{mb}^{\ell} \Big) \Big),
$$

$$
\Xi_{\mu_j}^r = \sum_{i \in \mathscr{N}_r} (1-\mu_i) \Big(\frac{\lambda_i}{\lambda_j} \sum_{m \in \mathscr{N}} \widetilde{\Omega}_{im}^{x} \, \widetilde{\psi}_{mj}^{x} + \psi_{ji}^{x} - \lambda_i \sum_{h \in \mathscr{H}} \frac{\psi_{jh}^{\ell}}{\Lambda_h} \Big(\mathbb{1}_{\{h \in \mathscr{H}_r\}} \widetilde{\Omega}_{ih}^{\ell} + \sum_{n \in \mathscr{N}} \widetilde{\Omega}_{in}^{x} \, \widetilde{\psi}_{nh}^{\ell} \Big) \Big),
$$

$$\Xi^r_{\widetilde{\Omega}^x_{jm}} = \sum_{i\in\mathcal{N}_r} (1-\mu_i)\Big(\psi^x_{mi} - \lambda_i \sum_{h\in\mathcal{H}} \frac{\psi^\ell_{mh}}{\Lambda_h}\Big(\mathbb{1}_{\{h\in\mathcal{H}_r\}}\,\widetilde{\Omega}^\ell_{ih} + \sum_{n\in\mathcal{N}} \widetilde{\Omega}^x_{in}\,\widetilde{\psi}^\ell_{nh}\Big)\Big),$$

$$\Xi^r_{\widetilde{\Omega}^\ell_{jh}} = \sum_{i\in\mathcal{N}_r} (1-\mu_i)\frac{\lambda_i}{\Lambda_h}\Big(\mathbb{1}_{\{h\in\mathcal{H}_r\}}\,\widetilde{\Omega}^\ell_{ih} + \sum_{n\in\mathcal{N}} \widetilde{\Omega}^x_{in}\,\widetilde{\psi}^\ell_{nh}\Big).$$

In particular, under *no cluster rents* $d\log Y_r = \sum_{i\in\mathcal{N}_r} \frac{\lambda_i}{\Phi_r}\, d\log A_i + \sum_{h\in\mathcal{H}_r} \frac{\Lambda_h}{\Phi_r}\, d\log L_h$.

The first decomposition in equation (7) isolates the sources of cluster-level real-value-added growth under *fully local factor markets*. The first term aggregates productivity shocks from all firms using the cluster-specific incidence weights $\ddot{\lambda}^r_i$. Thus, cluster r remains exposed to productivity changes originating both inside and outside the cluster through input-output linkages. The next terms show how local rents convert markup and factor-income-share changes into real-value-added effects. Increases in cluster-firm cost-to-GDP shares, $\mu_i\lambda_i$, raise Y_r with weight $(1-\mu_i)\lambda_i/\Phi_r$, the rent share of cluster-r value added generated by firm i. The propagation term $\sum_m \widetilde{\Omega}^x_{im} \sum_j \widetilde{\psi}^x_{mj}\, d\log\mu_j$ captures how markup shocks anywhere in the network affect cluster firms through downstream cost exposure. These rent-mediated effects are offset by changes in the factor-income distribution Λ: increases in factor-income shares for $h\in\mathcal{H}_r$ reduce Y_r in proportion to cluster firms' direct factor-cost exposure, $\widetilde{\Omega}^\ell_{ih}$, while $\sum_j \widetilde{\Omega}^x_{ij}\widetilde{\psi}^\ell_{jh}\, d\log\Lambda_h$ captures indirect exposure through intermediate-input linkages. The final term, $\sum_h \ddot{\Lambda}^r_h\, d\log L_h$, captures changes in local and external factor supplies, weighted by $\{\ddot{\Lambda}^r_h\}_{h\in\mathcal{H}}$.

The second decomposition in equation (8) rewrites the same first-order variation, $d\log Y_r$, in terms of distributional and network primitives. Substituting the first-order variations in λ and Λ yields a formula for how Y_r responds to changes in household expenditure shares χ_h, household final-demand shares β_{hj}, inverse markups μ_j, and firm cost shares on intermediate inputs and primary factors, $\widetilde{\Omega}^x_{jm}$ and $\widetilde{\Omega}^\ell_{jh}$. The coefficients $\Xi^r_{\chi_h}$, $\Xi^r_{\beta_{hj}}$, $\Xi^r_{\mu_j}$, $\Xi^r_{\widetilde{\Omega}^x_{jm}}$, and $\Xi^r_{\widetilde{\Omega}^\ell_{jh}}$ are sufficient statistics for the local effect of each primitive perturbation on cluster-level real value added under *fully local factor markets*. This representation separates the roles of technology, inverse-markup wedges, household demand composition, and firms' cost structure. It is therefore the more primitive decomposition: it shows not only that cluster-level real value added depends on rents and factor-income shares, but also which reallocations in household expenditure and firm input costs drive those dependencies.

The no-domestic-rents case provides the relevant Hulten-type benchmark. If firms in cluster r are competitive, $\mu_i = 1$ for all $i \in \mathcal{N}_r$, and *fully local factor markets* hold, cluster value added is exhausted by compensation to factors in $\mathcal{H}_r$, so the rent-mediated terms in Corollary 2 vanish. In that case, $d \log Y_r = \sum_{i \in \mathcal{N}_r} \frac{\lambda_i}{\Phi_r} d \log A_i + \sum_{h \in \mathcal{H}_r} \frac{\Lambda_h}{\Phi_r} d \log L_h$, so cluster real-value-added growth reduces to a Domar-weighted local productivity component plus a factor-income-weighted local factor-supply component. This is a cluster-level Hulten result: the local envelope condition eliminates first-order reallocation terms, leaving only productivity shocks to cluster firms and supply shocks to local factors. Unlike Hulten (1978), which applies to aggregate growth accounting in an efficient economy with intermediate inputs, this result applies to a production cluster embedded in a larger network. Under country-level factor-market segmentation, it specializes to the open-economy Hulten-type decompositions in Burstein & Cravino (2015) and Baqaee & Farhi (2024). The implication is that, at this benchmark, cluster TFP_r is first-order neutral to demand-side reallocation: expenditure-share shifts, markup perturbations around the competitive benchmark, and rent-mediated network propagation do not affect Y_r at first order, provided cluster value added is exhausted by local factor payments and factors do not move across clusters.

3.3 Relationship with Baqaee & Farhi (2024)

Baqaee & Farhi (2024) derive a decomposition of country-level real GDP Y_r in a general open-economy production network with arbitrary price-cost wedges. In their notation, the producers and factors located in country r are $\mathcal{N}_r$ and $\mathcal{F}_r$. Physical location and ownership need not coincide: firms and factors located in country r may generate income for households residing elsewhere. This section shows that Corollary 2 delivers the corresponding country-level specialization under *fully local factor markets*, but expresses it using incidence-based sufficient statistics. Relative to Baqaee & Farhi (2024), this representation has two practical advantages. First, it captures how productivity, markup, factor-income-share, and factor-supply shocks anywhere in the network affect Y_r through their incidence on cluster-r real value added. Second, it does not require changes in net imported quantities or their valuation shares in domestic GDP. More generally, Theorem 1 does not impose country-level factor-market segmentation, and therefore applies to arbitrary production clusters, including settings in which capital, intangible inputs, or digital labor can move across countries or clusters. The Online Appendix details the mapping between the two approaches.

The net quantity of good i produced in country-r is $q_{ri} \equiv \mathbb{1}_{\{i\in\mathcal{N}_r\}} y_i - \sum_{j\in\mathcal{N}_r} x_{ji}$. Its Divisia share in country-r GDP is therefore $\Omega_{Y_r i} \equiv p_i q_{ri}/GDP_r$. Accordingly, the country-r GDP deflator and real GDP satisfy $d\log p_{Y_r} = \sum_{i\in\mathcal{N}} \Omega_{Y_r i}\, d\log p_i$ and $d\log Y_r = \sum_{i\in\mathcal{N}} \Omega_{Y_r i}\, d\log q_{ri}$, where $d\log q_{ri}$ is interpreted as dq_{ri}/q_{ri} even when $q_{ri} < 0$. Let $\widehat{p}_{\mathcal{N}_r} \equiv d\log p_{\mathcal{N}_r}$, $\widehat{p}_{\mathcal{N}\setminus\mathcal{N}_r} \equiv d\log p_{\mathcal{N}\setminus\mathcal{N}_r}$, $\widehat{w}_{\mathcal{F}_r} \equiv d\log w_{\mathcal{F}_r}$, and $\widehat{A\mu}_{\mathcal{N}_r} \equiv d\log(A_{\mathcal{N}_r}\mu_{\mathcal{N}_r})$. Under the inverse-markup convention used here, domestic price variation is given by $\widehat{p}_{\mathcal{N}_r} = \left(I_{N_r} - \widetilde{\Omega}_x^{D_r}\right)^{-1}\left(\widetilde{\Omega}_\ell^r \widehat{w}_{\mathcal{F}_r} + \widetilde{\Omega}_x^{M_r}\widehat{p}_{\mathcal{N}\setminus\mathcal{N}_r} - \widehat{A\mu}_{\mathcal{N}_r}\right)$. Here $\widetilde{\Omega}_x^{D_r} \in \mathbb{R}^{N_r\times N_r}$ is the domestic cost-based input–output matrix, $\widetilde{\Omega}_x^{M_r} \in \mathbb{R}^{N_r\times(N-N_r)}$ is the imported cost-based input-output matrix, and $\widetilde{\Omega}_\ell^r \in \mathbb{R}^{N_r\times F_r}$ is the domestic cost-based factor-share matrix measuring country-r firms' exposure to local factor prices. The matrices $\widetilde{\Omega}_x^{D_r}$ and $\widetilde{\Omega}_x^{M_r}$ are obtained by restricting $\widetilde{\Omega}_x$ to rows $i \in \mathcal{N}_r$ and then partitioning columns into domestic suppliers, $j \in \mathcal{N}_r$, and foreign suppliers, $j \notin \mathcal{N}_r$; $\widetilde{\Omega}_\ell^r$ restricts $\widetilde{\Omega}_\ell$ to rows $i \in \mathcal{N}_r$ and local-factor columns $f \in \mathcal{F}_r$. Define $\widetilde{\Psi}_x^r \equiv (I_{N_r} - \widetilde{\Omega}_x^{D_r})^{-1}$, and let $\widetilde{\psi}_{ij}^{x_r}$ denote its (i,j) element for $i, j \in \mathcal{N}_r$. For $j \in \mathcal{N}_r$, define $\widetilde{\lambda}_{Y_r j} \equiv \sum_{i\in\mathcal{N}_r} \Omega_{Y_r i}\widetilde{\psi}_{ij}^{x_r}$. For $f \in \mathcal{F}_r$, define $\widetilde{\Lambda}_{Y_r f} \equiv \sum_{i\in\mathcal{N}_r} \Omega_{Y_r i} \sum_{j\in\mathcal{N}_r} \widetilde{\psi}_{ij}^{x_r}\widetilde{\Omega}_{jf}^{\ell}$. For $i \notin \mathcal{N}_r$, define $\widetilde{\Lambda}_{Y_r i} \equiv \sum_{m\in\mathcal{N}_r} \Omega_{Y_r m} \sum_{j\in\mathcal{N}_r} \widetilde{\psi}_{mj}^{x_r}\widetilde{\Omega}_{ji}^{x}$. Finally, define $\Lambda_{Y_r f} \equiv w_f L_f/GDP_r$ for $f \in \mathcal{F}_r$ and $\lambda_{Y_r i} \equiv p_i y_i/GDP_r$ for $i \in \mathcal{N}_r$.

Corollary 3. Baqaee & Farhi (2024). Under country-level factor-market segmentation, and defining $\Lambda_{Y_r i} \equiv -p_i q_{ri}/GDP_r$ for $i \notin \mathcal{N}_r$, country-r real GDP satisfies

$$
\begin{aligned}
d\log Y_r = &\sum_{j\in\mathcal{N}_r} \widetilde{\lambda}_{Y_r j}(d\log A_j + d\log \mu_j) - \sum_{f\in\mathcal{F}_r} \widetilde{\Lambda}_{Y_r f}\, d\log \Lambda_{Y_r f} \\
&+ \sum_{i\notin\mathcal{N}_r} (\widetilde{\Lambda}_{Y_r i} - \Lambda_{Y_r i})(d\log q_{ri} - d\log \Lambda_{Y_r i}) + \sum_{f\in\mathcal{F}_r} \widetilde{\Lambda}_{Y_r f}\, d\log L_f.
\end{aligned}
$$

Corollary 3 restates the Baqaee & Farhi (2024) first-order country-level real-GDP decomposition and makes explicit its connection to Corollary 2. Relative to Corollary 2, this representation is less convenient to implement in standard production-network data for three reasons. First, Corollary 2 captures productivity, markup, factor-income-share, and factor-supply shocks originating anywhere in the network through their incidence on cluster r. By contrast, in Corollary 3, the direct technology, markup, and factor-supply terms are attached to domestic firms and local factors; foreign-firm effects enter indirectly through the imported-net-quantity term involving $d\log q_{ri}$ and the import valuation shares $\Lambda_{Y_r i}$ in domestic GDP.

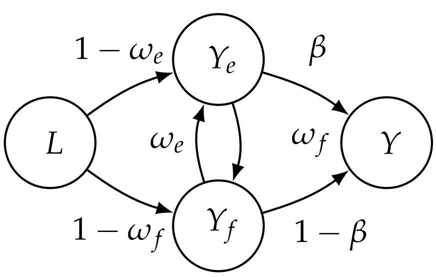
Second, Corollary 2 does not require tracking changes in imported net quantities. In the Baqaee & Farhi (2024) representation, implementation requires changes in the country-r net-output quantities q_{ri} for foreign goods $i \notin \mathcal{N}_r$—equivalently, changes in imported intermediate quantities $-q_{ri}$—together with the import valuation shares $\Lambda_{Y_r i} \equiv -p_i q_{ri}/GDP_r$. This distinction matters empirically because standard input-output tables usually report nominal flows. Recovering the corresponding quantity changes requires price or deflator information by origin, sector, and use, which is often unavailable or noisy. By contrast, the sufficient statistics in Corollary 2 are built from standard production-network observables: revenue shares, cost shares, expenditure shares, factor-income shares, and network centralities.

Finally, Theorem 1 is more general than the country-level real-GDP decomposition in Baqaee & Farhi (2024). Whereas Baqaee & Farhi (2024) assign producers and factors to countries and impose country-level factor-market segmentation, Theorem 1 imposes no analogous restriction on clusters. It therefore applies to arbitrary production clusters, not only countries, and accommodates settings in which factors move across clusters. These considerations make the theorem useful when clusters are not countries, or when factor mobility and missing imported-net-quantity data make Corollary 3 difficult to implement. When the cluster is a country and *fully local factor markets* hold, Corollary 2 provides the corresponding convenient implementation.

This distinction also matters for comovement. Huo et al. (2025) decompose international GDP comovement into network transmission and correlated shocks in a competitive production network. Their model is a useful benchmark, but it does not feature the rent-generating transmission channels emphasized here. Without wedges, there is no endogenous TFP-incidence channel operating through rents. Thus, their mechanism concerns country-level GDP comovement in a competitive network, whereas here, I emphasize cluster-level real value added and TFP in distorted networks.

3.4 Example 1: Two-Sector Circular Network with a Common Factor

Figure 1: Circular Network



L is a common factor used in both energy (e) and food (f) production. The two firms trade intermediates and operate CRS technologies $y_e = A_e\, Q_e(\ell_e, x_{ef})$ and $y_f = A_f\, Q_f(\ell_f, x_{fe})$, where ℓ_e and ℓ_f denote labor used in energy and food, x_{ef} is food used as an input in energy, and x_{fe} is energy used as an input in food. Define intermediate input cost shares

$\omega_e = p_f x_{ef}/\mu_e p_e y_e$ and $\omega_f = p_e x_{fe}/\mu_f p_f y_f$. A representative household supplies L, owns both firms, and consumes e and f through $Y = Q_Y(C_e, C_f)$, with energy expenditure share $\beta = p_e C_e/P_Y Y$. Nominal value added in sector $r \in \{e, f\}$ is $p_{Y_r} Y_r = (1 - \mu_r \omega_r) p_r y_r$. Applying Theorem 1 yields the following decompositions:

$$d\log Y_e = \ddot{\lambda}_e^e \, d\log A_e + \ddot{\lambda}_f^e \, d\log A_f + \left(\ddot{\lambda}_e^e + \lambda_e \left(\frac{\psi_{e|e} - \mu_e \omega_e}{\Phi_e} - \frac{\psi_e^\ell}{\Lambda} \right) \right) d\log \mu_e$$

$$+ \left(\ddot{\lambda}_f^e + \lambda_f \left(\frac{\psi_{f|e} + \mu_f \omega_f}{\Phi_e} - \frac{\psi_f^\ell}{\Lambda} \right) \right) d\log \mu_f + \mu_e \lambda_e \left(\frac{\psi_{f|e} - 1}{\Phi_e} + \frac{1 - \psi_f^\ell}{\Lambda} \right) d\omega_e$$

$$+ \mu_f \lambda_f \left(\frac{\psi_{e|e} + 1}{\Phi_e} + \frac{1 - \psi_e^\ell}{\Lambda} \right) d\omega_f + \left(\frac{\psi_{e|e} - \psi_{f|e} + 1}{\Phi_e} + \frac{\psi_f^\ell - \psi_e^\ell}{\Lambda} \right) d\beta + d\log L,$$

$$d\log Y_f = \ddot{\lambda}_e^f \, d\log A_e + \ddot{\lambda}_f^f \, d\log A_f + \left(\ddot{\lambda}_e^f + \lambda_e \left(\frac{\psi_{e|f} + \mu_e \omega_e}{\Phi_f} - \frac{\psi_e^\ell}{\Lambda} \right) \right) d\log \mu_e$$

$$+ \left(\ddot{\lambda}_f^f + \lambda_f \left(\frac{\psi_{f|f} - \mu_f \omega_f}{\Phi_f} - \frac{\psi_f^\ell}{\Lambda} \right) \right) d\log \mu_f + \mu_e \lambda_e \left(\frac{\psi_{f|f} + 1}{\Phi_f} + \frac{1 - \psi_f^\ell}{\Lambda} \right) d\omega_e$$

$$+ \mu_f \lambda_f \left(\frac{\psi_{e|f} - 1}{\Phi_f} + \frac{1 - \psi_e^\ell}{\Lambda} \right) d\omega_f + \left(\frac{\psi_{e|f} - \psi_{f|f} - 1}{\Phi_f} + \frac{\psi_f^\ell - \psi_e^\ell}{\Lambda} \right) d\beta + d\log L,$$

where

$$\lambda_e = \frac{\beta + (1-\beta)\mu_f \omega_f}{1 - \mu_e \omega_e \mu_f \omega_f}, \quad \lambda_f = \frac{1 - \beta + \beta \mu_e \omega_e}{1 - \mu_e \omega_e \mu_f \omega_f}, \quad \Lambda = (1 - \omega_e)\mu_e \lambda_e + (1 - \omega_f)\mu_f \lambda_f,$$

$$\Phi_e \equiv (1 - \mu_e \omega_e)\lambda_e, \qquad \Phi_f = (1 - \mu_f \omega_f)\lambda_f,$$

$$\phi_e \equiv (1 - \mu_e \omega_e)^{-1}(1 - \omega_e \omega_f)^{-1}, \qquad \phi_f \equiv (1 - \mu_f \omega_f)^{-1}(1 - \omega_e \omega_f)^{-1}$$

$$\psi_e^\ell = \frac{\mu_e(1 - \omega_e(1 - \mu_f(1 - \omega_f)))}{1 - \mu_e \omega_e \mu_f \omega_f}, \qquad \psi_f^\ell = \frac{\mu_f(1 - \omega_f(1 - \mu_e(1 - \omega_e)))}{1 - \mu_e \omega_e \mu_f \omega_f},$$

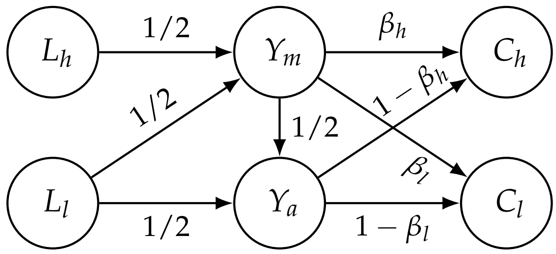
$$\ddot{\lambda}_e^e = \phi_e(1 - \mu_e \omega_e \omega_f), \quad \ddot{\lambda}_f^e = \phi_e(1 - \mu_e)\omega_e, \quad \ddot{\lambda}_f^f = \phi_f(1 - \mu_f \omega_f \omega_e), \quad \ddot{\lambda}_e^f = \phi_f(1 - \mu_f)\omega_f,$$

$$\psi_{f|e} = \frac{\mu_f \omega_f (1 - \mu_e \omega_e)}{1 - \mu_e \omega_e \mu_f \omega_f}, \quad \psi_{f|f} = -\psi_{f|e}, \quad \psi_{e|f} = \frac{\mu_e \omega_e (1 - \mu_f \omega_f)}{1 - \mu_e \omega_e \mu_f \omega_f}, \quad \psi_{e|e} = -\psi_{e|f}.$$

These expressions show that sectoral real value added responds to eight primitive perturbations in five classes. For each $r \in \{e, f\}$, $d\log Y_r$ responds to: (i) productivity in energy and food, $d\log A_e$ and $d\log A_f$; (ii) inverse-markup wedges in both firms, $d\log \mu_e$ and $d\log \mu_f$; (iii) intermediate-input intensities, $d\omega_e$ and $d\omega_f$; (iv) final-demand composition, $d\beta$; and (v) aggregate labor supply, $d\log L$. The coefficients are pinned down by equilibrium expenditure shares, wedges, and the circular network. Productivity shocks enter with rent-adjusted Domar weights $\ddot{\lambda}_s^r$, so a change in A_s affects Y_r both directly, when $s = r$, and indirectly, when $s \neq r$, through intermediate-input linkages and rent-adjusted pass-through. Inverse-markup shocks $d\log\mu_s$ affect Y_r through a direct rent-adjusted pass-through margin, governed by $\ddot{\lambda}^r_s$, and a reallocation margin, captured by the $\psi_{\cdot|r}$ and ψ^ℓ/Λ terms. Changes in input intensities, $d\omega_e$ and $d\omega_f$, and final demand, $d\beta$, affect both the cluster value-added share and distributional reallocation, changing the sensitivity of Y_r to upstream revenue pass-through and factor-income incidence. Finally, because L is the only primary factor, $d\log L$ enters one-for-one.

3.5 Example 2: Two sectors, Two households and a Common Factor

Figure 2: 2-Sector 2-Household Network



High-skill labor L_h is used only in manufacturing (m), while low-skill labor L_l is used in both manufacturing and agriculture (a). Firms operate CRS technologies $y_m = A_m Q_m(\ell_{mh}, \ell_{ml})$ and $y_a = A_a Q_a(\ell_{al}, x_{am})$, where ℓ_{mh} and ℓ_{ml} are high- and low-skill labor used in manufacturing, ℓ_{al} is low-skill labor used in agriculture, and x_{am} is manufacturing output, interpreted as machinery, used as an intermediate input in agriculture. Assume no wedges, $\mu_m = \mu_a = 1$, and symmetric fixed revenue shares: $\frac{1}{2} = \frac{w_h\ell_{mh}}{p_m y_m} = \frac{w_l\ell_{ml}}{p_m y_m} = \frac{w_l\ell_{al}}{p_a y_a} = \frac{p_m x_{am}}{p_a y_a}$. High- and low-skill households $b \in \{h, l\}$ each own one half of both firms, $\kappa_{mb} = \kappa_{ab} = 1/2$, supply their respective labor, and consume m and a through $C_b = Q^c_b(C_{bm}, C_{ba})$, with manufacturing expenditure share $\beta_b \equiv \frac{p_m C_{bm}}{p^c_b C_b}$. Nominal value added equals revenue in manufacturing, $p_{Y_m}Y_m = p_m y_m$, and equals revenue net of intermediate costs in agriculture, $p_{Y_a}Y_a = p_a y_a - p_m x_{am} = \frac{1}{2}p_a y_a$. Applying Theorem 1 with $\overline{\beta} \equiv (\beta_h + \beta_l)/2$ and $\Delta\beta \equiv \beta_h - \beta_l$ yields the following decompositions:

$$d\log Y_m = d\log A_m + \frac{1-\overline{\beta}}{3-\beta_h}\left(d\log\mu_m + \frac{3(1-\overline{\beta})}{1+\beta_l}d\log\mu_a\right) + \frac{1}{2}\left(d\log L_h + d\log L_l\right)$$
$$+\frac{1}{8}(4-\Delta\beta)\left(\frac{\Delta\beta(4-\Delta\beta)}{(1+\beta_l)(3-\beta_h)}d\chi_h + \frac{d\beta_h}{3-\beta_h} + \frac{d\beta_l}{1+\beta_l}\right),$$

$$d\log Y_a = 2\,d\log A_a - \frac{3(1-\overline{\beta})}{3-\beta_h}d\log\mu_a - \frac{1+\beta_l}{3-\beta_h}\,d\log\mu_m$$
$$-\frac{(4-\Delta\beta)}{8(1-\overline{\beta})}\left(\frac{(4-\Delta\beta)}{(3-\beta_h)}\Delta\beta\,d\chi_h + \frac{1+\beta_l}{(3-\beta_h)}d\beta_h + d\beta_l\right) + d\log L_l,$$

$$d\log Y = \frac{2(1+\beta_l)}{4-\Delta\beta}d\log A_m + \frac{4(1-\overline{\beta})}{4-\Delta\beta}d\log A_a + \frac{1+\beta_l}{4-\Delta\beta}d\log L_h + \frac{3-\beta_h}{4-\Delta\beta}d\log L_l.$$

The closed-form expressions highlight five forces. First, productivity shocks operate through direct sectoral technology channels: A_m affects Y_m, and A_a affects Y_a, with no cross-firm productivity exposure. This is the competitive-envelope implication: absent wedges, firm productivity does not transmit across firms through rent-adjusted pass-through. Second, inverse-markup shocks reallocate real value added across sectors but wash out in the aggregate. A higher markup in manufacturing or agriculture, equivalently a fall in μ_m or μ_a, lowers Y_m and raises Y_a. Thus, wedge shocks generate distributional movements in sectoral real value added rather than changes in aggregate efficiency. Third, expenditure composition affects only the sectoral split. A shift in expenditure toward the high-skill household, $d\chi_h > 0$, raises Y_m and lowers Y_a whenever $\beta_h > \beta_l$, since high-skill households tilt demand toward manufacturing. Fourth, an increase in either household's taste for manufacturing, higher β_h or β_l, also raises Y_m and lowers Y_a. Hence, demand recomposition through either (χ_h, χ_l) or (β_h, β_l) reallocates real value added across sectors but leaves aggregate real value added unchanged. Finally, labor-supply exposure differs sharply across sectors: manufacturing loads on both L_h and L_l with elasticity $1/2$, whereas agriculture loads only on low-skill labor.

4 Sectoral TFP Incidence in the U.S. Production Network

This section implements Theorem 1 in U.S. sectoral data. I construct sector-level TFP growth for 66 private industries over 1997–2021 by combining BEA production-network data, BLS productivity data, OEWS occupation-level labor income, and CEX income-specific consumption shares. The implementation captures both production-network linkages and household heterogeneity: occupations differ in industry exposure, and households differ in consumption baskets. I use the resulting sectoral TFP series to study aggregate TFP contributions, sectoral comovement, and the sectoral anatomy of the Great Financial Crisis.

4.1 Data

The empirical implementation follows the data construction in Rojas-Bernal (2026) and combines BEA, BLS, OEWS, and CEX sources. BEA's Industry-Level Production Accounts with Extended Capital Detail provide observed industry productivity growth, $d \log A_{i,t}$, and capital compensation by nine asset classes; I harmonize them to the Production Accounts classification and build a balanced industry-asset-year panel for

1997–2021. BEA Use and Make tables for 1997–2021, restricted to 66 private industries and excluding government and scrap/used categories, provide the production network. Row-normalizing yields the industry input-output matrix. For each industry-year, I construct sales, value added, and input costs: labor, intermediates, and asset-level capital compensation. I measure inverse markups as μ = Cost/Sales, where Cost = Labor + Capital + Intermediates, and construct industry-year productivity shocks from ILPA. I use OEWS data to construct occupation o's labor cost share in industry i, $\widetilde{\Omega}^{\ell}_{io} = \left(\frac{\textit{Labor Income}_{io,t}}{\sum_{o'} \textit{Labor Income}_{io',t}}\right)\left(\frac{\textit{Labor Costs}_{i,t}}{\textit{Cost}_{i,t}}\right)$, and stack 833 occupations with nine capital assets to form $\widetilde{\Omega}_{\ell}$. Finally, I use CEX to map income to 66-industry expenditure shares by averaging spending within each household-year's 100 nearest-income neighbors. Let $\bar{J}_{o,t}$ denote mean OEWS earnings for occupation o, and set $\beta_{o,t}$ equal to the CEX expenditure-share vector of the household whose income is closest to $\bar{J}_{o,t}$. To match Production Accounts final demand β^{IO}_t, I define the residual non-labor bundle as $\beta^{NL}_t = (1 - \sum_o \Lambda_{o,t})^{-1} \left(\beta^{IO}_t - \sum_o \Lambda_{o,t}\beta_{o,t}\right)$.

The model has 66 sectors and 843 household types: 833 labor-income households, one for each occupation o; nine capital-income households, one for each asset class; and one residual household that receives profits. Labor-income households consume according to $\beta_{o,t}$, capital-income and profit-earning households consume according to β^{NL}_t. Under this specification, model-implied aggregate TFP closely tracks BLS aggregate TFP growth, with an index of agreement of 0.91 and a correlation of 0.87.

To make the equilibrium objects concrete, Figure 3 reports the 2019 sector-by-factor matrix $\ddot{\Lambda}^r_h$, where each cell measures the incidence of L_h on Y_r. The columns aggregate the 833 occupations into 22 SOC major groups and add the nine capital types. The heatmap shows that factor exposure is not sector-exclusive: many sectors load on management-related and office-administrative occupations, as well as on structures on the capital side. At the same time, natural sector-specific patterns appear: health-care sectors load heavily on health occupations, while housing and other real-estate sectors are strongly exposed to structures. The heatmap displays one high-dimensional production-side incidence matrix, which is later collapsed into partial-equilibrium sufficient statistics and compared with full general-equilibrium responses in Section 5.

Implementing Theorem 1 delivers annual sector-level TFP growth rates. I use these series to study realized outcomes: which sectors drive the aggregate TFP path, which sectors comove with the rest of the economy, and which recover or remain scarred during the GFC. In Section 5, I compare partial-equilibrium sufficient statistics with full general-equilibrium adjustments under a parametric nested-CES structure.

4.2 Sectoral Contributions to Aggregate TFP Levels

I first measure each sector's cumulative contribution to the aggregate TFP path. The baseline index cumulates sectoral TFP growth, $d \log TFP_t = \sum_{i \in \mathcal{N}} \Phi_{i,t-1} \, d \log TFP_{i,t}$. For each sector i, I construct a leave-one-out path that omits sector i's contribution and renormalizes the remaining value-added weights. The terminal gap between the baseline and leave-one-out paths measures sector i's cumulative contribution.

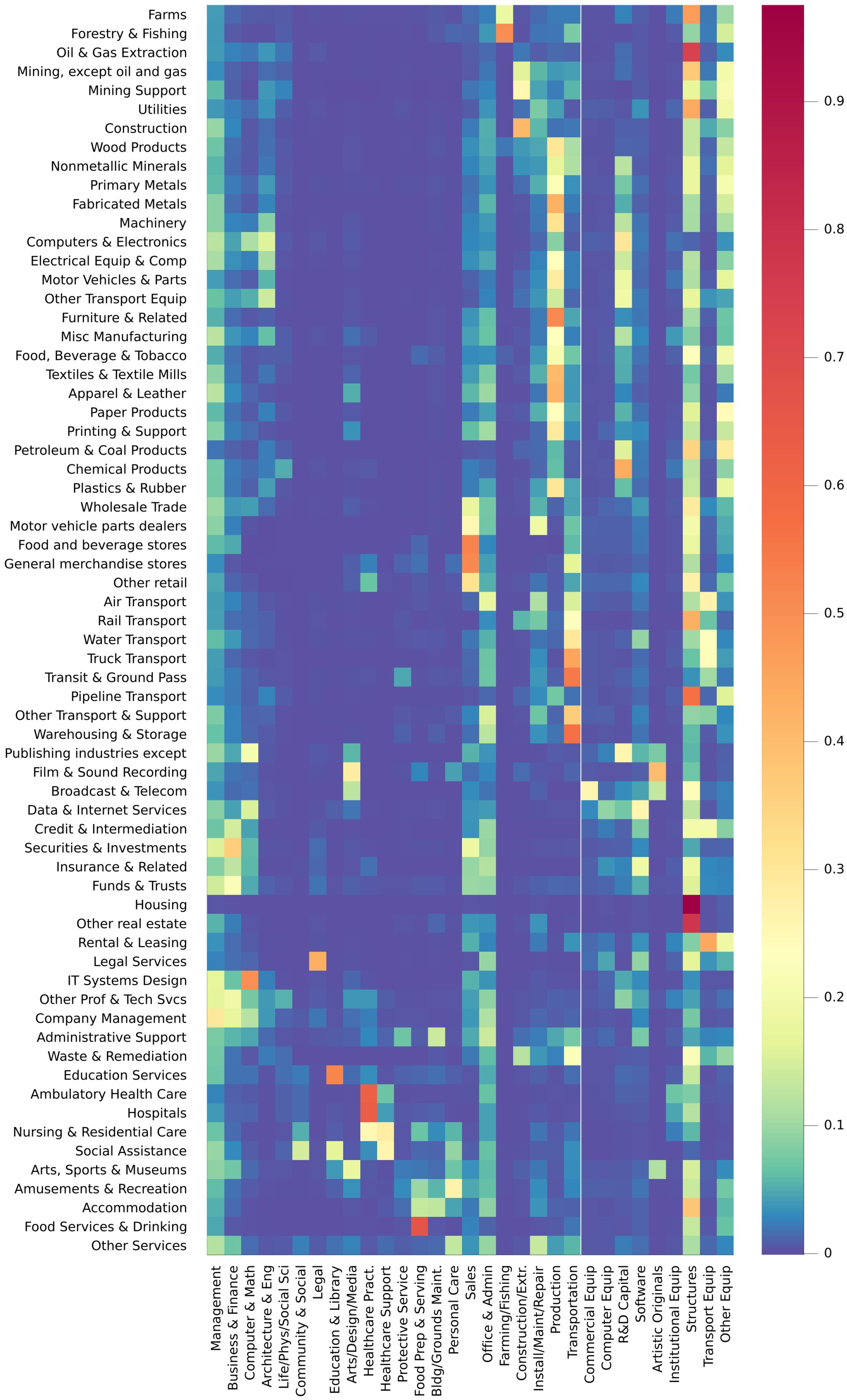
Table 4: Top 10 Sectors with Largest Positive and Negative Effect on TFP

Sector	Diff	Sector	Diff
Computers & Electronics	4.21%	Farms	-4.44%
Oil & Gas Extraction	2.89%	Construction	-2.38%
IT Systems Design	2.76%	Housing	-2.11%
Administrative Support	2.59%	Other Services	-1.84%
Data & Internet Services	1.94%	Chemical Products	-1.71%
Broadcast & Telecom	1.29%	Rental & Leasing	-1.35%
Ambulatory Health Care	1.27%	Credit & Intermediation	-1.15%
Other real estate	1.23%	Fabricated Metals	-1.14%
Insurance & Related	1.09%	Education Services	-0.79%
Publishing industries except	0.87%	Apparel & Leather	-0.76%

Note: Diff is the 2021 terminal gap between the baseline aggregate TFP path and the leave-one-out path excluding sector i. The baseline cumulates $d \log TFP_t = \sum_{i \in \mathcal{N}} \Phi_{i,t-1} d \log TFP_{i,t}$ from 1997=100; the leave-one-out path renormalizes remaining value-added weights. Positive values mean exclusion lowers terminal TFP; negative values mean exclusion raises it. Sample: 66 BEA industries, 1997–2021.

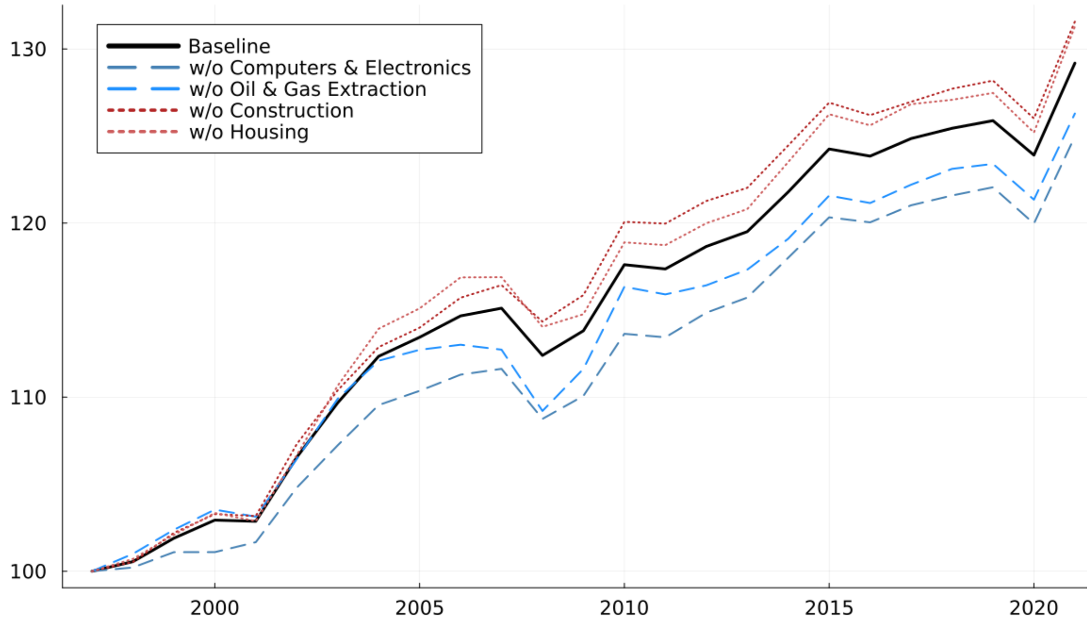
Table 4 and Figure 4 summarize sectoral contributions to aggregate TFP over the full sample. Computers and electronics, oil and gas extraction, and administrative support are the largest positive contributors, while farms, construction, and housing are the largest negative contributors. Aggregate TFP is therefore shaped by a small set of cumulative contributors and drags rather than by smooth contributions from all sectors. The gaps are time-varying rather than parallel, indicating that sectoral contributions are concentrated in particular episodes. This motivates separating full-sample level effects from the sectoral anatomy of specific episodes, especially the GFC contraction and subsequent recovery.

Figure 3: Y_r Elasticities $\ddot{\Lambda}^r_h$, 2019



Note: Cell (r, h) reports the elasticity of Y_r to factor-h supply. Columns group 833 occupations into 22 SOC major groups, with 9 capital types to the right of the white line.

Figure 4: Counterfactual TFP Level Paths Excluding Individual Sectors, 1997–2021



Note: The solid black line is the baseline aggregate TFP path, cumulating $d \log TFP_t = \sum_{i \in \mathcal{N}} \Phi_{i,t-1} d \log TFP_{i,t}$ from 1997=100. Dashed/dotted lines are leave-one-out paths, $\Delta \log TFP_t^{-i} = \sum_{j \neq i} \frac{\Phi_{j,t-1}}{\sum_{m \neq i} \Phi_{m,t-1}} \Delta \log TFP_{j,t}$. Blue dashed lines exclude the two largest positive contributors; red dotted lines exclude the second- and third-largest negative contributors.

4.3 Sectoral TFP Comovement and Network Position

I next measure comovement with the rest of the economy. Figure 5 plots, for each sector, the correlation between sector-level and aggregate TFP growth over 1998–2021. Open markers report the baseline, $\rho_i = \text{corr}(d \log TFP_{i,t}, d \log TFP_t)$, where the aggregate index includes sector i. Filled markers report the leave-one-out correlation, $\rho_i^{-i} = \text{corr}(d \log TFP_{i,t}, d \log TFP_t^{-i})$, where aggregate TFP is recomputed excluding sector i and renormalizing the remaining value-added weights. Sectors are ordered by ρ_i^{-i}.

Table 5 relates comovement to sector size and network position. Larger sectors exhibit more negative leave-one-out comovement under both revenue- and cost-based Domar weights. The cost-based weight $\widetilde{\lambda}_i = (\widetilde{\Psi}'_x \beta' \chi)_i$ measures the share of aggregate value added that passes through sector i through downstream cost linkages. The coefficients on both λ_i and $\widetilde{\lambda}_i$ are negative and statistically significant at the 1% level in the ρ_i^{-i} regressions, while neither measure explains the baseline correlation ρ_i. Thus, sectors that are large in sales or central to other sectors' cost structures tend to move against the rest of the economy's TFP.

The *downstream extraction* wedge $\widetilde{\lambda}_i - \lambda_i$ also predicts lower leave-one-out comovement at the 5% level. This object measures cost-based centrality in excess of revenue centrality: it is large when a sector is more important as an upstream input into other sectors' cost structures than its own sales would suggest. The negative coefficient implies that upstream, cost-central sectors have TFP growth that is more negatively correlated with the rest of the economy. Thus, network position matters beyond size: sectors whose cost-based centrality exceeds their revenue centrality systematically counter-move with the rest-of-economy TFP component.

Finally, the *customer Herfindahl index*, $HHI_i = \sum_{j \in \mathcal{N}} \left(\lambda_j W_{ji} / \sum_{k \in \mathcal{N}} \lambda_k W_{ki} \right)^2$, where $W_{ji} = p_i x_{ji} / p_j^x X_j$ is buyer j's within-intermediate-input share sourced from supplier i, measures the concentration of sector i's downstream customer base in the input-output network, weighting buyers by Domar weights. This statistic enters negatively and is significant at the 10% level. Sectors selling to diversified customer bases, with low HHI_i, are exposed to a broad cross-section of downstream shocks and therefore comove more strongly with the leave-one-out aggregate. By contrast, sectors selling to a few large buyers load more heavily on buyer-specific shocks, pulling their comovement with the rest of the economy toward zero or negative values.

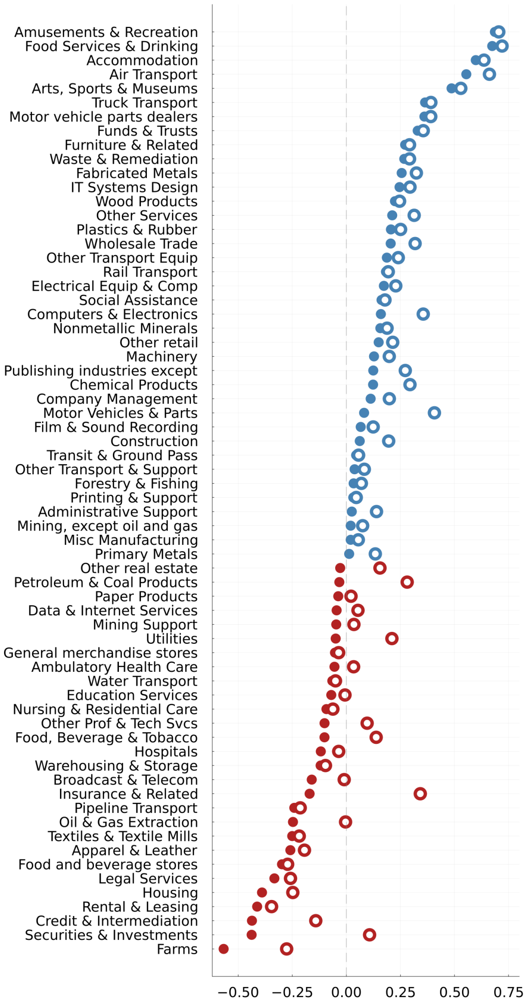
Taken together, these results show that network position shapes sectoral comovement along two margins. First, sectors that are large in Domar terms, or whose cost-based centrality exceeds their revenue centrality, tend to move against the rest-of-economy TFP component. Second, sectors with concentrated downstream customer bases are less synchronized with the aggregate because their TFP growth loads more on buyer-specific shocks than on broad economy-wide movements.[2]

4.4 The GFC and the Sectoral Anatomy of Recovery

Figure 6 summarizes the sectoral anatomy of the Great Financial Crisis. The left panel reports cumulative TFP changes for three windows: the 2006–2007 pre-stress period, the 2008–2009 core crisis, and the 2006–2009 stress-and-crisis window. The right panel then reports the 2010–2012 recovery for the same sectors, ordered by cumulative change in

[2]These patterns do not appear for a broader set of unreported sectoral regressors. At conventional significance levels, neither ρ_i nor ρ_i^{-i} is significantly related to alternative measures of network position, size, volatility, or sector type: $\widetilde{\lambda}_i / \lambda_i$, standard upstreamness and downstreamness, final-demand exposure, markups, value-added shares, sector-level TFP volatility, mean and mean absolute sector-level TFP contributions, a goods-producing-sector dummy, row and column sums of $\ddot{\lambda}$, sums of Ψ_ℓ, Leontief output multipliers, and granular-volatility measures such as $\sigma_i \lambda_i$, $\sigma_i \widetilde{\lambda}_i$, and $\lambda_i \sigma_i / \sqrt{\sum_j \sigma_j^2}$.

Figure 5: Sectoral TFP Correlation with Aggregate TFP: ρ_i and ρ_i^{-i}



Note: Open dots report ρ_i over 1998–2021; filled dots report ρ_i^{-i}.

Table 5: Cross-sectional regressions of sectoral comovement

	ρ_i	ρ_i^{-i}
Panel A		
λ_i	-0.138	-2.662^{**}
	(1.044)	(1.104)
Cons	0.151^{***}	0.122^{***}
	(0.043)	(0.045)
R^2	0.000	0.083
Panel B		
$\widetilde{\lambda}_i$	-0.516	-2.973^{***}
	(1.048)	(1.093)
Cons	0.164^{***}	0.135^{***}
	(0.045)	(0.047)
R^2	0.004	0.104
Panel C		
$\widetilde{\lambda}_i - \lambda_i$	-3.739	-38.982^{**}
	(17.995)	(19.283)
Cons	0.152^{***}	0.102^{**}
	(0.040)	(0.043)
R^2	0.001	0.060
Panel D		
HHI_i	-0.255	-0.397^{*}
	(0.190)	(0.207)
Cons	0.188^{***}	0.108^{**}
	(0.042)	(0.046)
R^2	0.027	0.054
N	66	

Note: Regressors are evaluated in 1997. $^{***}p < 0.01$, $^{**}p < 0.05$, $^{*}p < 0.1$.

TFP during the 2006–2009 stress-and-crisis window. The figure shows that the crisis was not a uniform sectoral TFP collapse. The largest stress-and-crisis window (2006–2009) declines were concentrated in motor vehicles and parts (-74.18%), funds and trusts (-73.31%), furniture (-41.26%), petroleum and coal products (-40.23%), nonmetallic minerals (-37.93%), apparel and leather (-30.05%), other transportation and support (-27.18%), construction (-24.71%), mining except oil and gas (-24.65%), wood products (-24.03%), fabricated metals (-22.39%), textiles (-20.29%), and primary metals (-19.38%).

The timing of these declines is central. Motor vehicles did not exhibit pre-crisis TFP stress: it rose 9.79% in 2006–2007 and then collapsed by 76.48% in 2008–2009. Funds and trusts, by contrast, is the cleanest pre-crisis collapse: it fell 38.25% before the recession and another 56.77% during 2008–2009, for a cumulative 73.31% stress-and-crisis window. Construction, mining except oil and gas, petroleum and coal, textiles, and other transportation and support also declined in both the pre-stress and core-crisis windows. These sectors are therefore better examples of persistent deterioration over the stress-and-crisis window, rather than a one-time recession shock.

The recovery panel in Figure 6 shows that the post-crisis rebound was highly uneven. Some large losers recovered sharply: motor vehicles rose 730.08% in 2010–2012, funds and trusts rose 70.77%, apparel and leather 68.25%, primary metals 49.36%, wood products 40.50%, and fabricated metals 15.31%. The motor-vehicles rebound is so large that it is capped in the recovery panel for readability. But this rebound pattern was not general. Several sectors continued to lose TFP after the crisis window.

Figure 7 plots the 2010–2012 recovery against the 2006–2009 full-cycle contraction. The dashed 45-degree line marks a recovery exactly equal to the earlier loss. The fitted lines are much flatter, indicating that sectors with larger crisis losses tended to recover more, but not enough on average to undo the contraction; this is especially clear for the value-added-weighted fit, which gives more weight to larger sectors. Motor vehicles and parts is excluded from the fits because its rebound is an extreme outlier. Several sectors show no recovery, with negative TFP changes both in 2006–2009 and in 2010–2012: petroleum and coal products fell 40.23% during 2006–2009 and another 28.18% during the recovery; mining except oil and gas fell 24.65% and 27.64%; other transportation and support, 27.18% and 7.26%; construction, 24.71% and 6.56%; textiles, 20.29% and 4.26%; and other services, 14.26% and 4.75%.

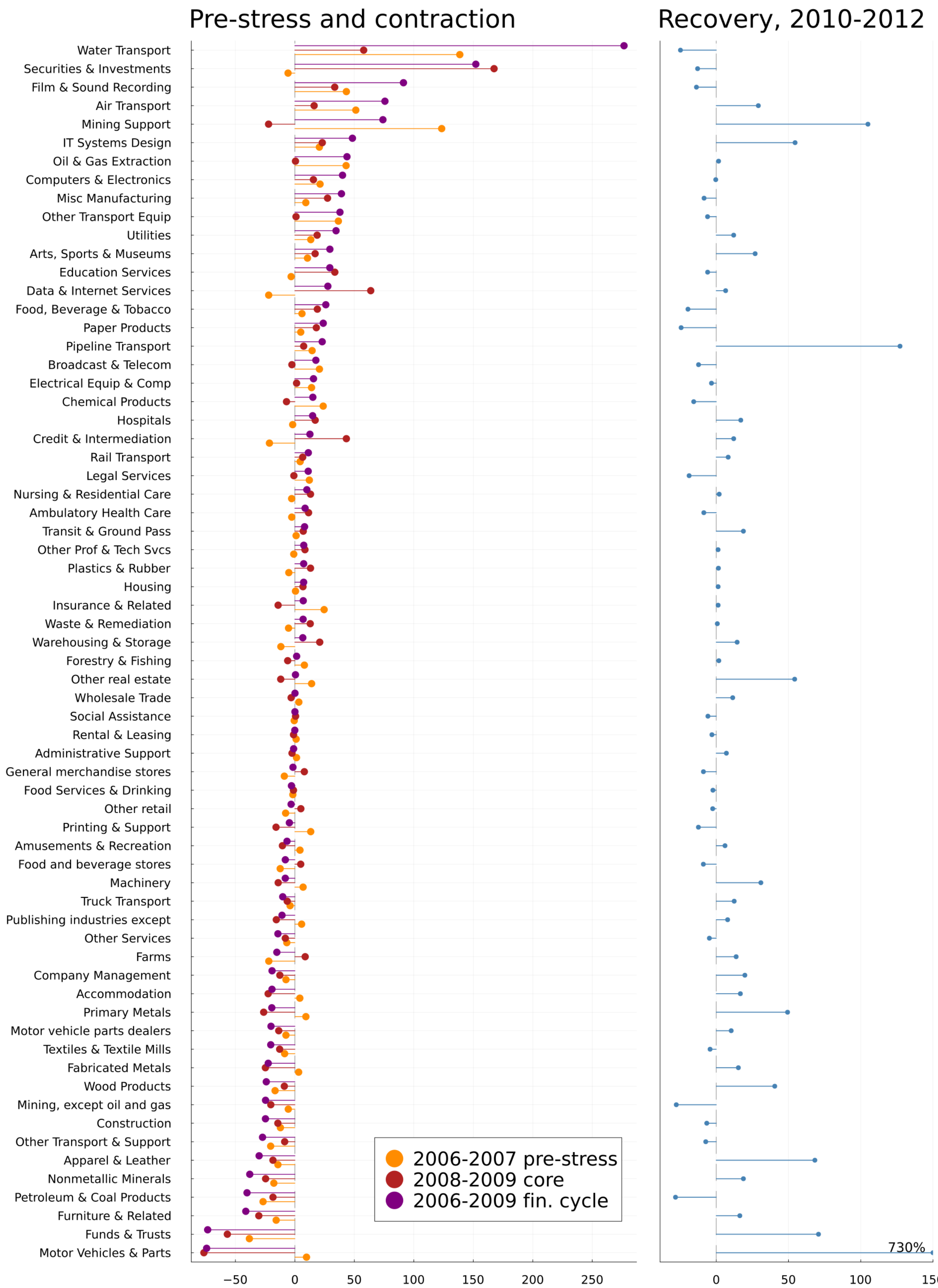
Table 6: Financial-Sector TFP Across Great Financial Crisis Windows

Sector	2006–2007	2008–2009	2006–2009
Credit & Intermediation	-21.38%	43.29%	12.66%
Securities & Investments	-5.74%	167.47%	152.11%
Insurance & Related	24.6%	-14.08%	7.05%
Funds & Trusts	-38.25%	-56.77%	-73.31%
Rental & Leasing	1.06%	-1.11%	-0.07%

Note: Entries equal $100[\exp(\sum_t \Delta \log TFP_{i,t}) - 1]$, so they are cumulative level changes, not sums of annual percentage changes. Positive values indicate TFP increases; negative values indicate declines.

Table 6 isolates financial and asset-related sectors and shows that "finance" was not homogeneous. Funds and trusts is the clear collapse case: TFP falls in both 2006–2007 and 2008–2009, for a cumulative 73.31% decline over 2006–2009. Credit intermediation

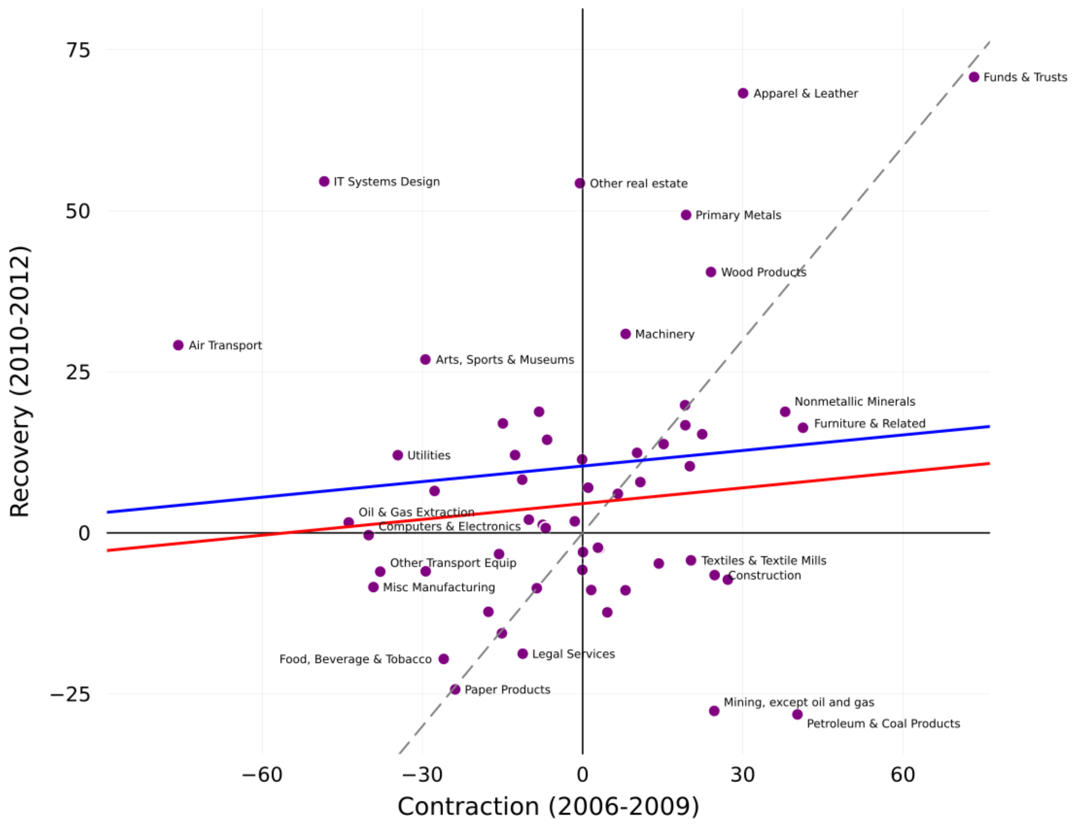
Figure 6: Sectoral TFP Around the Great Financial Crisis



Note: The left panel reports the 2006–2007 pre-crisis stress, 2008–2009 core crisis, and 2006–2009 full stress-and-crisis window; the right panel reports the 2010–2012 recovery. Sectors are ordered by cumulative 2006–2009 TFP change. Changes equal $100[\exp(\sum_t \Delta \log TFP_{i,t}) - 1]$, so they are cumulative level changes. Recovery values are capped at 150% for readability; capped observations are plotted at the boundary and labeled with actual values.

falls sharply in the pre-stress period (-21.38%) but rises 43.29% during 2008–2009, leaving a positive 12.66% change over the full stress-and-crisis window. Securities and investments is even more unusual: after a small pre-stress decline (-5.74%), TFP rises 167.47% during 2008–2009 and ends the full 2006–2009 window up 152.11%. Insurance falls in the core crisis (-14.08%), but its pre-crisis gain leaves it positive over 2006–2009.

Figure 7: Sectoral Contraction and Recovery Around the Great Financial Crisis



Note: Sectoral TFP recovery in 2010–2012 versus cumulative 2006–2009 losses. Horizontal-axis contractions equal $100[1 - \exp(\sum_t \Delta \log TFP_{i,t})]$; recovery equals $100[\exp(\sum_t \Delta \log TFP_{i,t}) - 1]$. The dashed line is 45 degrees. The blue fit is unweighted; the red fit is weighted by initial value-added shares, $\Phi_{i,2006}$.

The preceding figures describe the collapse and early recovery; Table 7 extends the horizon by measuring medium-run scarring, defined as sectoral TFP in 2014 relative to 2007. The scarring pattern is not identical to the 2006–2009 contraction ranking. Some sectors that looked damaged during the crisis continued to show persistent losses: mining except oil and gas is the largest scarred sector, with TFP 52.02% below its 2007 level; other transportation and support remains 20.23% below 2007; construction 17.48% below; rental and leasing 11.82% below; and other services 10.32% below. At the same time, some crisis-period losers ended up with large medium-run gains. Funds and trusts, despite its severe 2006–2009 collapse, is 120.06% above its 2007 TFP level by

2014; motor vehicles and parts is 81.41% above; and apparel and leather is 58.45% above. This reinforces the message from Figure 7: the crisis generated both scarring and reallocation, and the sectors with the largest short-run losses were not necessarily those with the largest medium-run scars.

Table 7: Largest Sectoral TFP Losses and Gains from 2007 to 2014

Largest 2014 TFP Scars Relative to 2007		Largest 2014 TFP Gains Relative to 2007	
Sector	Level change	Sector	Level change
Mining, except oil and gas	-52.02	Pipeline Transport	158.99
Chemical Products	-28.14	Data & Internet Serv.	130.82
Printing & Support	-26.08	Funds & Trusts	120.06
Legal Services	-23.95	Mining Support	115.98
Fabricated Metals	-21.86	IT Systems Design	90.19
Other Transport & Supp.	-20.23	Motor Vehicles & Parts	81.41
Construction	-17.48	Securities & Invest.	73.7
Rental & Leasing	-11.82	Air Transport	65.38
Other Services	-10.32	Arts, Sports & Museums	61.44
Broadcast & Telecom	-6.6	Apparel & Leather	58.45

Note: The table reports the sectors with the largest negative and positive TFP level changes between 2007 and 2014. Sectoral TFP levels are normalized to 100 in 2007 and evolved forward using sectoral TFP growth rates. The left panel lists the sectors with the largest remaining TFP losses by 2014, while the right panel lists the sectors with the largest TFP gains relative to 2007. Entries are index-point changes relative to the 2007 sectoral TFP level. Negative values indicate persistent TFP scarring, while positive values indicate that the sector's TFP level exceeded its 2007 value by 2014.

Table 8 formalizes the distinction between short-run recovery and medium-run scarring. Columns (1)–(3) use 2010–2012 TFP recovery as the dependent variable, while column (4) uses the 2014 TFP level relative to 2007. The table separates the core 2008–2009 contraction from the broader 2006–2009 contraction. The recovery regressions show bounce-back from the core crisis: in column (2), a one percentage point larger 2008–2009 TFP decline predicts a 1.106 percentage point larger 2010–2012 recovery, significant at the 1% level. The coefficient on the broader 2006–2009 contraction is smaller, 0.452, and significant at the 10% level, consistent with that window mixing temporary collapse with more persistent deterioration.

Medium-run scarring is selective. Customer concentration is the strongest pre-crisis structural predictor. Customer HHI is positive and significant at the 10% level in the recovery regressions in columns (1) and (3), and positive and significant at the 1% level in the scarring regression, with a coefficient of 131.31. Since the scarring outcome is the 2014 TFP level relative to 2007, this implies that sectors with more concentrated customer bases experienced smaller medium-run scars or larger medium-run gains. The result suggests that customer structure matters for post-crisis TFP dynamics. The other significant medium-run result is the goods-sector coefficient: goods-producing sectors are 25.59 percentage points lower relative to 2007, significant at the 5% level, consistent with persistent losses in mining, chemicals, fabricated metals, construction, and transportation support. By contrast, pre-crisis sector size, λ, and the wedge $\widetilde{\lambda} - \lambda$ are not statistically significant.

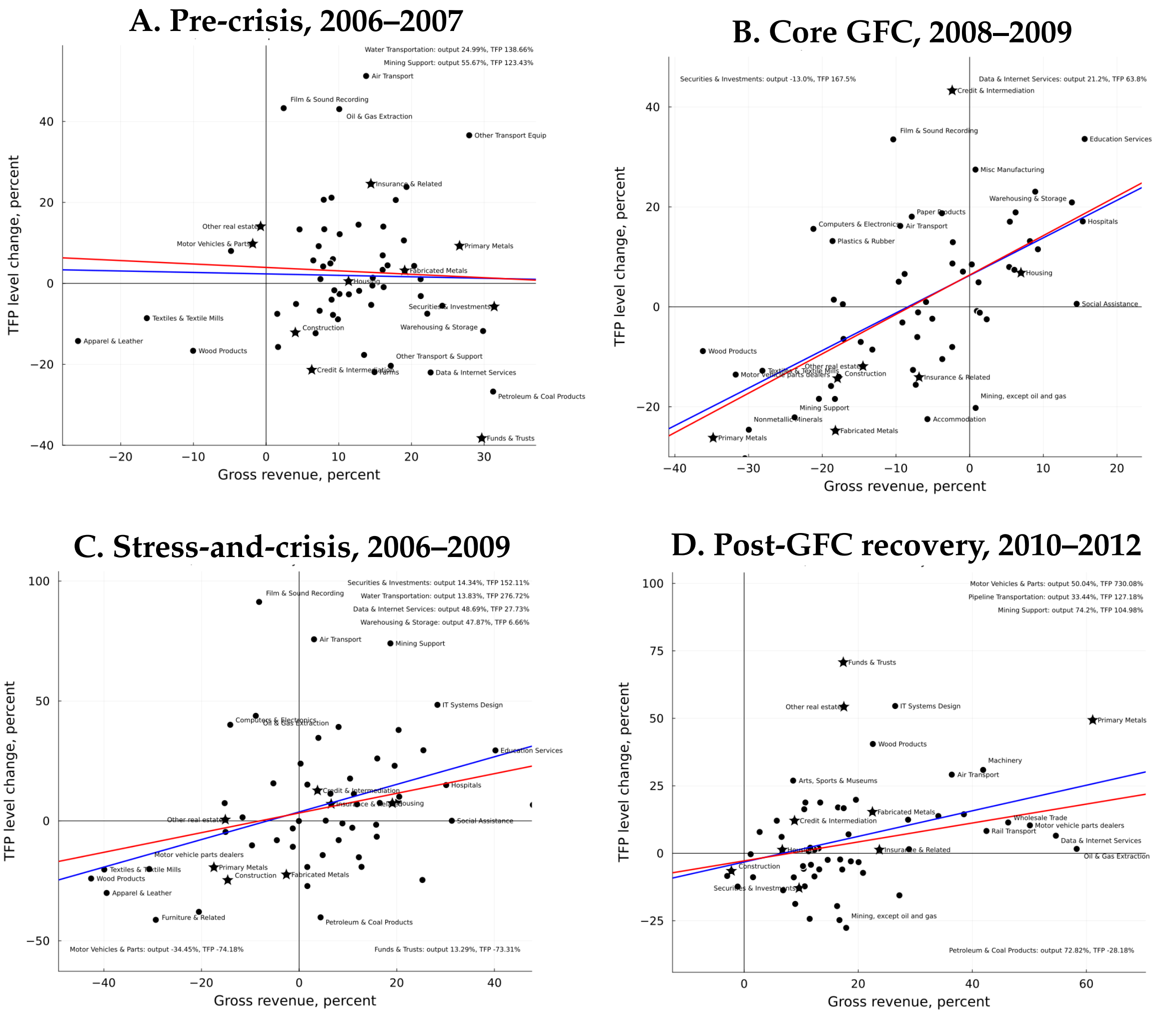
Table 8: Sectoral Recovery and Scarring After the Great Financial Crisis

	(1) Recovery	(2) Recovery	(3) Recovery	(4) Scarring
Core		1.106***		
		(0.368)		
Extended			0.452*	
			(0.232)	
λ	415.581	619.232	388.352	−320.362
	(534.007)	(506.634)	(522.33)	(198.702)
$\tilde{\lambda} - \lambda$	−7871.438	−10245.218	−8284.225	2056.496
	(9932.687)	(9372.08)	(9714.324)	(3695.914)
Customer HHI	127.267*	95.888	131.016*	131.31***
	(72.504)	(68.965)	(70.919)	(26.978)
Goods sector	19.706	4.858	12.84	−25.586**
	(26.207)	(25.132)	(25.866)	(9.752)
Cons	−12.012	−0.908	−4.349	12.602*
	(19.68)	(18.869)	(19.64)	(7.323)
R^2	0.102	0.22	0.156	0.307
N	66	66	66	66

Note: Cross-sectional regressions of sectoral TFP recovery and medium-run scarring after the GFC. Columns (1)–(3) use 2010–2012 TFP level changes; column (4) uses 2014 TFP relative to 2007. Core contraction is the 2008–2009 TFP decline; extended contraction is the 2006–2009 decline. All TFP changes are level changes. The scarring regression excludes core and extended because they are part of the post-2007 TFP path. Controls are pre-crisis sector size λ, the expenditure-revenue wedge $\widetilde{\lambda} - \lambda$, customer concentration, and a goods-sector indicator; λ, $\widetilde{\lambda}$, and customer concentration are measured in 2007.

Figure 8 compares gross revenue growth with sectoral TFP growth across the same GFC windows. The figure separates scale from efficiency: revenue captures the size of sectoral activity, while TFP captures the efficiency with which real value added is generated. The quadrants organize the interpretation. The upper-right quadrant contains sectors with rising revenue and TFP; the lower-left contains sectors with falling revenue and TFP. The lower-right is especially important for the crisis build-up: revenue expands while TFP falls, suggesting expansion without corresponding efficiency gains.

Figure 8: Gross Revenue and TFP Around the Great Financial Crisis



Note: Panels plot cumulative sectoral revenue growth against cumulative sectoral TFP growth. Changes are level percent changes, $100[\exp(\sum_t \Delta \log x_{i,t}) - 1]$. Zero lines separate revenue and TFP increases from declines. Star markers indicate finance-, housing-, construction-, auto-, and metals-related sectors emphasized in the GFC discussion. Blue lines show unweighted linear fits; red lines show fits weighted by pre-window value-added shares, Φ_i. Off-panel labels report sectors outside the displayed range.

The upper-left captures the opposite case: revenue contracts while measured TFP rises, consistent with downsizing, selection, or exit of low-productivity activity.

The pre-crisis period, 2006–2007, is unusual because the revenue–TFP relationship is weak and slightly negative. Several crisis-relevant sectors are in the lower-right quadrant: credit intermediation, securities and investments, funds and trusts, construction, data and internet services, and petroleum and coal products, all of which see revenue growth while TFP declines. During the core crisis, 2008–2009, the relationship becomes strongly positive, reflecting broad contraction in both activity and productivity. Construction, fabricated metals, primary metals, insurance, accommodation, textiles, and motor vehicle parts dealers fall in both revenue and TFP. At the same time, securities and investments and credit intermediation move into the upper-left quadrant: revenue falls while measured TFP rises, consistent with downsizing or selection. Over the 2006–2009 stress-and-crisis window, the mixed nature of the episode is clearest. Motor vehicles and parts, construction, metals, textiles, wood products, apparel and leather, and furniture show conventional contraction, while funds and trusts and petroleum and coal products combine rising revenue with large declines in TFP. Funds and trusts is the strongest case: revenue rises 13.29%, while TFP falls 73.31%.

The recovery window, 2010–2012, restores a positive revenue–TFP relationship for many sectors, but recovery remains uneven. Motor vehicles and parts is the most extreme rebound, with revenue rising 50.04% and TFP rising 730.08%; pipeline transportation and mining support also post large TFP gains. Yet petroleum and coal products and mining, excluding oil and gas, see revenue growth while TFP falls. The main message of Figure 8 is that sectoral TFP is not a proxy for sales, revenue, output, or scale. It captures the efficiency with which real value added is generated. This distinction matters for interpreting the GFC: in the build-up to the crisis, several important sectors were expanding in revenue while measured TFP deteriorated, suggesting inefficient expansion or overextension; during the core crisis, some sectors contracted in revenue while measured TFP improved, consistent with downsizing or selection; and during the recovery, most sectors returned to a positive revenue–TFP relationship, but important exceptions remained. Thus, sectoral TFP reveals stress and adjustment margins that revenue growth alone would miss.

5 General Equilibrium Propagation

I now turn to counterfactual incidence. The goal is to assess when the partial-equilibrium sufficient statistics in Corollary 1 approximate full general-equilibrium responses. For each shock, I compare the local sufficient statistic with the response under a normalized nested-CES model. This comparison identifies when the statistics are reliable guides to equilibrium incidence and when endogenous changes in expenditure shares, factor-income shares, and cost shares materially change the answer.

Following Baqaee & Farhi (2024) and Rojas-Bernal (2026), let $\dot{z} \equiv z/\bar{z}$ denote a variable normalized by its equilibrium level. Firm $i \in \mathcal{N}$ produces using the normalized nested CES: $\dot{y}_i = A_i\big((1-\omega_i^x)\dot{L}_i^{\frac{\theta-1}{\theta}} + \omega_i^x \dot{X}_i^{\frac{\theta-1}{\theta}}\big)^{\frac{\theta}{\theta-1}}$, $\dot{L}_i = \big(\sum_{h\in\mathscr{H}} \alpha_{ih}\dot{\ell}_{ih}^{\frac{\theta_\ell-1}{\theta_\ell}}\big)^{\frac{\theta_\ell}{\theta_\ell-1}}$, $\dot{X}_i = \big(\sum_{j\in\mathscr{N}} W_{ij}\dot{x}_{ij}^{\frac{\theta_x-1}{\theta_x}}\big)^{\frac{\theta_x}{\theta_x-1}}$. Here, θ is the elasticity of substitution between primary factors and intermediate inputs, θ^ℓ is the elasticity within the primary-factor nest, and θ^x is the elasticity within the intermediate-input nest. Similarly, each household h consumes a normalized CES aggregate $\dot{C}_h = \big(\sum_{i\in\mathscr{N}} \beta_{hi}\dot{C}_{hi}^{\frac{\varrho-1}{\varrho}}\big)^{\frac{\varrho}{\varrho-1}}$,where ϱ is the elasticity of substitution across final goods. The key advantage of the normalized CES model is that ω_i^ℓ, α_{ih}, ω_i^x, W_{ij}, and β_{hi} retain their interpretation from Section 2: they are equilibrium cost or expenditure shares. I parameterize the model using the best-fit elasticities from Rojas-Bernal (2026), imposing common elasticities across sectors and households: $\theta = 0.001$, $\theta_\ell = 0.946$, $\theta_x = 2.319$, and $\varrho = 0.065$. Thus, the calibrated model features near-Leontief substitution between primary factors and intermediates, approximately Cobb–Douglas substitution within the primary-factor nest, high substitutability across intermediate inputs, and strong complementarity across final consumption goods.

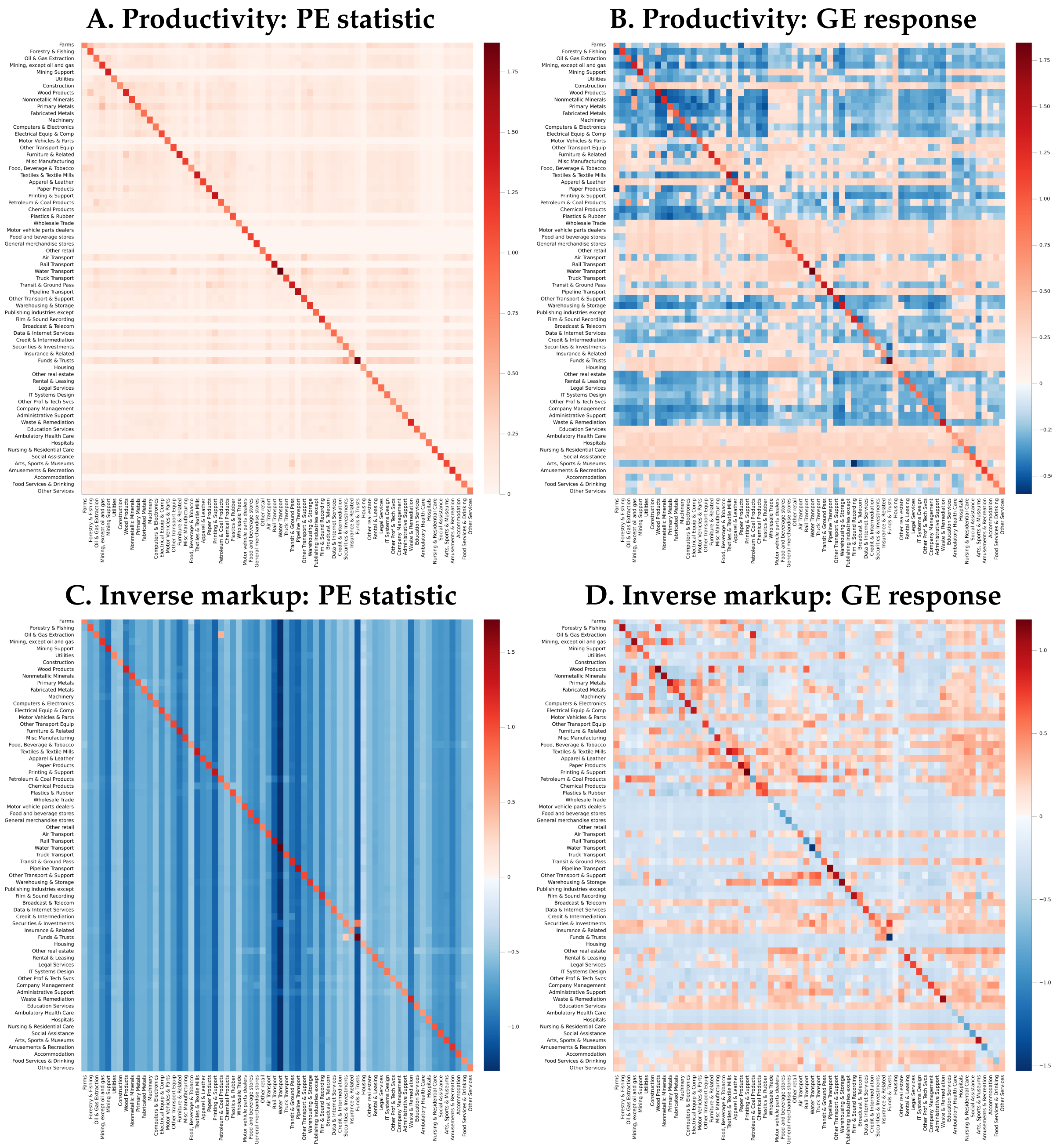
5.1 Sectoral Shock Incidence: PE versus GE

Figure 9 compares the partial-equilibrium sufficient statistics from Corollary 1 with the corresponding general-equilibrium responses. For sectoral productivity shocks, Panels A and B normalize each column so the aggregate technology component equals 1%.[3] The PE matrix is positive everywhere and strongly diagonal: productivity shocks raise TFP in the shocked sector, with positive spillovers elsewhere. The GE matrix preserves this structure but adds dispersion and some negative off-diagonal entries. These reflect reallocation margins absent from the PE statistic: once expenditure shares and network linkages adjust, productivity gains in one sector can lower measured TFP in others. Still, the PE statistic closely tracks the GE response, with a correlation of 0.975 and an index of agreement of 0.987.

Markup shocks behave very differently. Panels C and D consider increases in μ, equivalently markup reductions, normalized so the aggregate competitiveness component equals 1%. In the PE statistic, diagonal entries are positive: reducing a sector's markup raises TFP in that sector. Off-diagonal entries are largely negative, so the PE cross-sectoral incidence of markup reductions is mostly adverse for other sectors. The GE effect substantially changes this pattern. Some diagonal entries become neg-

[3] A shock to sector i is scaled by $1/\widetilde{\lambda}_i$, so sectors with different baseline centrality are compared as shocks with the same direct aggregate technology effect.

Figure 9: Sectoral TFP Incidence: Sufficient Statistics and GE Effects



Note: Panels A and C report the PE sufficient statistics from Corollary 1; Panels B and D report the corresponding GE responses. Panels A–B use technology shocks; Panels C–D use markup shocks. Each cell (r, i) gives the effect on sector-r TFP of a shock to sector i. Technology shocks are scaled so the direct aggregate technology component equals 1%; inverse-markup shocks are scaled so the direct aggregate competitiveness component equals 1%. Heatmaps plot $\text{sign}(x)|x|^{1/3}$ to preserve signs while compressing magnitudes. Warmer colors are positive; blue cells are negative.

ative, meaning that a markup reduction need not benefit the shocked sector. Some off-diagonal entries become positive, showing that cross-sectoral allocative-efficiency gains. Quantitatively, the PE statistic is a poor approximation: the PE and GE matrices have a correlation of -0.172 and an index of agreement of only 0.097.

The main lesson is that partial-equilibrium sufficient statistics work well for productivity shocks but not for markup shocks. For productivity shocks, the direct sectoral incidence matrix already captures most of the full general-equilibrium response. For markup shocks, by contrast, equilibrium reallocation is central, so the partial-equilibrium statistic misses the sign and magnitude of many sectoral effects.

5.2 Sufficient Statistic Targeting: Expenditure Reallocation

Following Rojas-Bernal (2026), this subsection uses expenditure sufficient statistics to target budget-neutral lump-sum transfers. The basic rule ranks occupations by the relevant statistic and transfers $10,000 per worker from the upper half to the lower half of the distribution. The progressive rule uses the same ranking but matches low-statistic recipients with higher-statistic donors, prioritizing lower-income recipients and higher-income donors; unmatched workers receive zero. Transfers are weighted by 2019 OEWS employment and scaled by 2019 GDP.

Figure 10 reports the sectoral TFP effects. Panel A targets aggregate TFP using the aggregate expenditure statistic: the aggregate response is small, 0.0052% under the basic rule and 0.0014% under the progressive rule, but the sectoral responses are much larger. Under the basic rule, TFP rises most in food, beverage, and tobacco (0.90%), farms (0.79%), miscellaneous manufacturing (0.43%), and motor vehicles and parts (0.39%), while it falls most in truck transportation (-1.03%), and nursing and residential care (-0.87%). Panel B targets computer and electronic products using the sector-specific statistic from Corollary 1, $\Phi_r^{-1}(\beta_{h|r} + \mathscr{B}_{h|r}) - M_h^r$: the aggregate effects again remain small, 0.0048% under the basic rule and 0.0033% under the progressive rule, but the largest response is not in the targeted sector, since motor vehicles and parts rises by 2.66%, compared with 0.39% for computer and electronic products. Thus, expenditure targeting generates small aggregate TFP effects through sizable sectoral reallocations.

6 Conclusion

These findings underscore a principle: real value added is an incidence object. A firm-level shock changes not only aggregate productivity but also the real value added of other firms. I provide a first-order decomposition for that object. For any firm cluster, real-value-added growth splits into a factor-supply component and a TFP component, with TFP further decomposed into technology, competitiveness, and distributional

Figure 10: Sectoral TFP Responses to Targeted Lump-Sum Transfers

Note: First-order sectoral log-TFP responses (percent) to budget-neutral $10,000-per-worker transfers in 2019. Transfers are assigned by occupation and weighted by 2019 OEWS employment. The basic scheme assigns $+1/-1$ to workers below/above the median relevant statistic; the progressive scheme matches high-statistic donors with lower-income, low-statistic recipients and assigns 0 to unmatched workers. Panel A ranks households by aggregate M_h; Panel B by $\Phi_r^{-1}(\beta_{h|r} + \mathscr{B}_{h|r}) - M_h^r$ for r = computer and electronics, which orders households by their pairwise PE effect on the target by Corollary 1. Filled/hollow markers denote basic/progressive transfers; blue/red markers denote positive/negative responses. Computed around the 2019 equilibrium.

reallocation. The sufficient statistics are not aggregate Domar weights. They are cluster-specific incidence weights that measure how firms, factors, households, wedges and expenditure patterns affect the real value added of a particular cluster. Hulten's benchmark is recovered only when the cluster has no rents and fully local factor markets. Away from that benchmark, markups, ownership, demand composition and network reallocations become first-order determinants of cluster-level real value added.

The empirical results show that this distinction matters. In U.S. sectoral data from 1997–2021, aggregate TFP is shaped by a small set of positive and negative sectoral contributors: computers and electronics, and oil and gas extraction, are major positive contributors, while farms, construction, and housing are major drags. Level contributions and comovement are distinct objects: large Domar sectors and upstream cost-central sectors tend to comove negatively with the rest of the economy, while sectors with concentrated downstream customer bases tend to decouple from the aggregate. The GFC further illustrates the value of the TFP measure. Several financial-cycle sectors expanded revenue before the crisis, while measured TFP deteriorated, revealing overexpansion that revenue alone would miss. During the crisis, some financial sectors contracted revenue while TFP rose, consistent with downsizing or selection. Finally, the counterfactual exercises show that the partial-equilibrium sufficient statistics are reliable guides for productivity shocks, but not for markup shocks, whose incidence depends critically on full general-equilibrium adjustment.

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Online Appendix

1 Proofs for the nonparametric model

1.1 Firms

Firm $i \in \mathscr{N}$ chooses $\{\ell_{ih}\}_{h\in\mathscr{H}}$, $\{x_{ij}\}_{j\in\mathscr{N}}$ to $\min \bar{c}_i = p_i^{\ell} L_i + p_i^x X_i = \sum_{h\in\mathscr{H}} w_h \ell_{ih} + \sum_{j\in\mathscr{N}} p_j x_{ij}$ subject to $1 = A_i\, Q_i(L_i, X_i)$, $L_i = A_i^{\ell}\, Q_i^{\ell}(\{A_{ih}^{\ell}\, \ell_{ih}\}_{h\in\mathscr{H}})$, $X_i = A_i^x Q_i^x(\{A_{ij}^x\, x_{ij}\}_{j\in\mathscr{N}})$, $mc_i = \mu_i\, p_i$, taking prices $\{w_h\}_{h\in\mathscr{H}}$ and $\{p_j\}_{j\in\mathscr{N}}$ as given. The optimality conditions for L_i and X_i are: $p_i^{\ell} = \varkappa_i\, A_i \frac{\partial\, Q_i(L_i, X_i)}{\partial\, L_i}$ and $p_i^x = \varkappa_i\, A_i \frac{\partial\, Q_i(L_i, X_i)}{\partial\, X_i}$, where $\varkappa_i$ is the Lagrange multiplier. Multiplying both conditions by L_i and X_i and summing, we obtain $\mu_i\, p_i = p_i^{\ell}\, L_i + p_i^x\, X_i = \varkappa_i\, A_i\, Q_i(L_i, X_i) = \varkappa_i$. The first-order conditions are given by

$$p_i^{\ell} = \mu_i\, p_i\, A_i\, \frac{\partial\, Q_i(L_i,\, X_i)}{\partial\, L_i}, \quad \text{if} \quad \frac{\partial\, Q_i(L_i,\, X_i)}{\partial\, L_i} > 0, \tag{9}$$

$$p_i^{x} = \mu_i\, p_i\, A_i\, \frac{\partial\, Q_i(L_i,\, X_i)}{\partial\, X_i}, \quad \text{if} \quad \frac{\partial\, Q_i(L_i,\, X_i)}{\partial\, X_i} > 0, \tag{10}$$

$$w_h = \mu_i\, p_i\, A_i\, \frac{\partial\, Q_i(L_i,\, X_i)}{\partial\, L_i}\, A_i^{\ell}\, \frac{\partial\, Q_i^{\ell}(\{A_{ib}^{\ell}\, \ell_{ib}\}_{b\in\mathscr{H}})}{\partial\, \ell_{ih}} \quad \text{if} \quad \frac{\partial\, y_i}{\partial\, \ell_{ih}} > 0, \tag{11}$$

$$p_j = \mu_i\, p_i\, A_i\, \frac{\partial\, Q_i(L_i,\, X_i)}{\partial\, X_i}\, A_i^{x}\, \frac{\partial\, Q_i^{x}(\{A_{im}^{x}\, x_{im}\}_{m\in\mathscr{N}})}{\partial\, x_{ij}} \quad \text{if} \quad \frac{\partial\, y_i}{\partial\, x_{ij}} > 0. \tag{12}$$

Let $e(a, b) = (\partial a/\partial b)(b/a)$. Then: $\omega_i^{\ell} = e(y_i, L_i) = \frac{1}{\mu_i} \frac{p_i^{\ell} L_i}{p_i y_i}$, $\omega_i^x = e(y_i, X_i) = \frac{1}{\mu_i} \frac{p_i^x X_i}{p_i y_i}$, $\alpha_{ih} = e(L_i, \ell_{ih}) = \frac{w_h \ell_{ih}}{p_i^{\ell} L_i}$, $\omega_{ij} = e(X_i, x_{ij}) = \frac{p_j x_{ij}}{p_i^x X_i}$, $\widetilde{\Omega}_{ih}^{\ell} = e(y_i, \ell_{ih})$, and $\widetilde{\Omega}_{ij}^x = e(y_i, x_{ij})$.

1.2 Households' Problem

Household $h \in \mathscr{H}$ chooses $\{\{C_{hi}\}_{i\in\mathscr{N}},\, L_h\}$ to maximize $U_h(C_h, L_h)$ subject to $C_h = Q_h^c(\{C_{hi}\}_{i\in\mathscr{N}})$ and the budget constraint $E_h = p_h^c\, C_h = \sum_{i\in\mathscr{N}} p_i\, C_{hi} \leq w_h\, L_h + \sum_{i\in\mathscr{N}} \kappa_{ih}\, \pi_i + T_h$. The household takes as given $\{w_h,\, T_h,\, \{p_i,\, \kappa_{ih},\, \pi_i\}_{i\in\mathscr{N}}\}$.

The first-order conditions for household $h \in \mathscr{H}$ are given by

$$\frac{w_h}{p_h^c} = -\frac{U_{L_h}}{U_{C_h}}, \tag{13}$$

$$\frac{p_i}{p_h^c} = \frac{\partial\, C_h}{\partial\, C_{hi}} \quad \text{if} \quad \frac{\partial\, C_h}{\partial C_{hi}} > 0. \tag{14}$$

It follows that $\beta_{hi} = e(C_h,\, C_{hi}) = \frac{p_i C_{hi}}{p_h^c C_h}$. Substitutabilities are captured by $e(C_h, L_h) = \frac{w_h L_h}{p_h^c C_h}$, and cross-elasticities by $e(C_{hi}, C_{hm}) + \frac{p_m C_{hm}}{p_i C_{hi}} = 0$.

1.3 General Equilibrium Conditions

Proposition 2. The allocation and price system are an equilibrium iff markets clear, fiscal budgets balance, and the following conditions are jointly satisfied:

$$\frac{\partial C_h}{\partial C_{hj}} \Big/ \frac{\partial C_h}{\partial C_{hi}} = \mu_i \frac{\partial y_i}{\partial x_{ij}} \quad \forall i,j \in \mathscr{N} : \ C_{hi} > 0, \ C_{hj} > 0, \ x_{ij} > 0, \tag{15}$$

$$-\frac{w_b}{w_h} \frac{U_{L_h}}{U_{C_{hi}}} = \mu_i \frac{\partial y_i}{\partial \ell_{ib}} \quad \forall i \in \mathscr{N}, \ \forall h,b \in \mathscr{H} : \ C_{hi} > 0, \ \ell_{ib} > 0, \ U_{L_h} < 0. \tag{16}$$

Necessity and sufficiency are proven in Rojas-Bernal (2026).

1.4 Equilibrium Centralities

Goods market. Substituting (10), (12), and (14) into market clearing for firm $i \in \mathscr{N}$

$$S_i = \sum_{h \in \mathscr{H}} p_i C_{hi} + \sum_{j \in \mathscr{N}} p_i x_{ji} = \sum_{h \in \mathscr{H}} \beta_{hi} E_h + \sum_{j \in \mathscr{N}} \Omega^x_{ji} S_j, \tag{17}$$

where $\Omega^x_{ij} \equiv \mu_i \, \omega^x_i \, \omega_{ij}$. In matrix form $(I_N - \widetilde{\Omega}'_x \, diag(\mu))S = \beta' E$ or $S = \mathscr{B}' E$, where $S \equiv [S_1, \cdots, S_N]'$, $E \equiv [E_1, \cdots, E_H]'$, and $\mu \equiv [\mu_1, \cdots, \mu_N]'$. Then $\lambda = \mathscr{B}' \chi$.

Define the cost-based consumer-to-firm matrix $\widetilde{\mathscr{B}} \equiv \beta \, \widetilde{\Psi}_x \equiv \beta \, \Psi_x (I_N - \Omega_x) \widetilde{\Psi}_x \equiv \mathscr{B}(I_N - \Omega_x)\widetilde{\Psi}_x$, where $\widetilde{\Psi}_x \equiv (I_N - \widetilde{\Omega}_x)^{-1}$. By construction, $\lambda = \Psi'_x \left(I_N - \widetilde{\Omega}'_x \right) \widetilde{\mathscr{B}}' \chi$, which motivates the definition of the cost-based Domar weights $\widetilde{\lambda} \equiv \widetilde{\Psi}'_x \, (I_N - \Omega'_x) \, \lambda \equiv \widetilde{\mathscr{B}}' \chi$. Notice $\widetilde{S}_i \equiv \sum_{h \in \mathscr{H}} p_i \, C_{hi} + \sum_{j \in \mathscr{N}} \widetilde{\Omega}^x_{ji} \, \widetilde{S}_j = \sum_{h \in \mathscr{H}} \widetilde{\mathscr{B}}_{hi} \, E_h$, where $\widetilde{S}_i = \widetilde{\lambda}_i \, GDP$. In equilibrium, $\widetilde{\Omega}^x_{ji}$ is the share of firm j's cost spent on intermediate inputs supplied by firm i. Hence $\widetilde{S}_i$ is the total value (gross) value added that passes through firm i, and, for a given expenditure distribution χ, $\widetilde{\lambda}_i$ is the share of aggregate value added that flows through i. Moreover, the identity $\omega'_\ell \, \widetilde{\lambda} = 1$ implies that $\omega^\ell_i \, \widetilde{\lambda}_i$ is the share of aggregate value added generated by primary factors employed at firm i. Finally, as shown in Rojas-Bernal (2026): $\widetilde{\Psi}_x \succcurlyeq \Psi_x$, $\widetilde{\mathscr{B}} \succcurlyeq \mathscr{B}$, and $\widetilde{\lambda} \succcurlyeq \lambda$.

Factor markets. Substituting (9) and (11) into the market clearing condition for factor $h \in \mathscr{H}$ yields $J_h = w_h \, L_h = \sum_{i \in \mathscr{N}} w_h \, \ell_{ih} = \sum_{i \in \mathscr{N}} \mu_i \, \omega^\ell_i \, \alpha_{ih} \, S_i$, where $\widetilde{\Omega}^\ell_{ih} \equiv \omega^\ell_i \, \alpha_{ih}$. In matrix form $J = \widetilde{\Omega}'_\ell \, diag(\mu) \, S = \Omega'_\ell \, S$, where $J \equiv [J_1, \cdots, J_H]'$. Dividing by nominal GDP yields $\Lambda = \Omega'_\ell \, \lambda$. Similarly, the cost-based factor weights are given by $\widetilde{\Lambda} \equiv \widetilde{\Omega}'_\ell \, \widetilde{\lambda}$, where $\mathbb{1}'_H \, \widetilde{\Lambda} = \mathbb{1}'_H \, \alpha' \, diag(\omega_\ell) \, \widetilde{\lambda} = \omega'_\ell \, \widetilde{\lambda} = 1$. Consequently $\widetilde{\Lambda} \succcurlyeq \Lambda$. The firm-to-factor and factor-to-firm centrality matrices are respectively given by $\Psi_\ell = \Psi_x \, \Omega_\ell$ and $\widetilde{\Psi}_\ell = \widetilde{\Psi}_x \, \widetilde{\Omega}_\ell$, where $\widetilde{\Psi}_\ell \, \mathbb{1}_H = \widetilde{\Psi}_x \, \widetilde{\Omega}_\ell \, \mathbb{1}_H = \widetilde{\Psi}_x \, \omega_\ell = \widetilde{\Psi}_x \left(I_N - \widetilde{\Omega}_x \right) \mathbb{1}_N = \mathbb{1}_N$. Additionally $\widetilde{\Psi}_\ell \succcurlyeq \Psi_\ell$. Similarly, the consumer-to-factor and factor-to-consumer centrality matrices are respectively given by $\mathscr{C} = \mathscr{B}\,\Omega_\ell$ and $\widetilde{\mathscr{C}} = \widetilde{\mathscr{B}}\,\widetilde{\Omega}_\ell$, where $\widetilde{\mathscr{C}}\,\mathbb{1}_H = \widetilde{\mathscr{B}}\,\widetilde{\Omega}_\ell\,\mathbb{1}_H = \beta\,\widetilde{\Psi}_x\,\omega_\ell = \beta\,\widetilde{\Psi}_x\left(I_N - \widetilde{\Omega}_x\right)\mathbb{1}_N = \mathbb{1}_H$, $\widetilde{\mathscr{C}}'\,\chi = \widetilde{\Omega}_\ell'\,\widetilde{\mathscr{B}}'\,\chi = \widetilde{\Omega}_\ell'\,\widetilde{\lambda} = \widetilde{\Lambda}$, $\mathscr{C}'\,\chi = \Omega_\ell'\,\mathscr{B}'\chi = \Omega_\ell'\,\lambda = \Lambda$, and $\widetilde{\mathscr{C}} \succcurlyeq \mathscr{C}$. Proof for the inequalities can be found in Rojas-Bernal (2026).

Household budget. Using $mc_i = \mu_i\,p_i$, firm costs are given $\bar{c}_i = \mu_i\,S_i$, and profits are $\pi_i = (1-\mu_i)S_i$. Applying (11), $E_h = \sum_{i\in\mathscr{N}}(\mu_i\,\omega_i^\ell\,\alpha_{ih} + \kappa_{ih}(1-\mu_i))p_i\,y_i + T_h = \sum_{i\in\mathscr{N}}(\mu_i\,\widetilde{\Omega}_{ih}^\ell + \kappa_{ih}(1-\mu_i))S_i + T_h$. In matrix form, $E = (\Omega_\ell + \Omega_\pi)'S + T$, where $T = [T_1, \cdots, T_H]'$ and $\Omega_\pi = diag(\mathbb{1}_N - \mu)\,\kappa$. Dividing by nominal GDP $\chi = (\Omega_\ell + \Omega_\pi)'\lambda + \tau$, where $\tau = [\tau_1, \cdots, \tau_H]'$ and $\tau_h = T_h/GDP$. It follows that $\mathbb{1}_H'(\Omega_\ell + \Omega_\pi)'\lambda = \mathbb{1}_H'\,\Lambda + \mathbb{1}_N'\,diag(\mathbb{1}_N - \mu)\,\lambda = \sum_{h\in\mathscr{H}}\Lambda_h + \sum_{i\in\mathscr{N}}(1-\mu_i)\lambda_i = 1$.

Nominal GDP. Start by aggregating revenue for $\mathscr{N}_r$. Let the associated set of households be $\mathscr{H}_r$. Then $\sum\limits_{i\in\mathscr{N}_r} S_i = \underbrace{\sum\limits_{i\in\mathscr{N}_r} p_i \sum\limits_{h\in\mathscr{H}_r} C_{hi}}_{=E_r^{dom}} + \underbrace{\sum\limits_{i\in\mathscr{N}_r}\Big(\sum\limits_{h\notin\mathscr{H}_r} p_i C_{hi} + \sum\limits_{j\notin\mathscr{N}_r} p_i\,x_{ji}\Big)}_{=Exp_r} + \overbrace{\sum\limits_{i\in\mathscr{N}_r}\sum\limits_{j\in\mathscr{N}_r} p_i\,x_{ji}}^{=Intra_r}$. E_r^{dom} denotes resident-household expenditure on goods produced by $\mathscr{N}_r$. Exp_r denotes exports: final demand by non-residents and intermediate demand by firms outside $\mathscr{N}_r$. $Intra_r$ denotes within-cluster intermediate sales. The role of $\mathscr{H}_r$ is purely notational: it segments revenue into internal sales and exports. Nominal imports for $\mathscr{N}_r$ are $Imp_r = \underbrace{\sum\limits_{j\notin\mathscr{N}_r} p_j \sum\limits_{h\in\mathscr{H}_r} C_{hj}}_{=E_r^{for}} + \underbrace{\sum\limits_{j\notin\mathscr{N}_r} p_j \sum\limits_{i\in\mathscr{N}_r} x_{ij}}_{=Imp_r^{Int}}$. Here E_r^{for} denotes resident-household expenditure on goods produced outside $\mathscr{N}_r$ (final-good imports), and Imp_r^{Int} denotes intermediate-input purchases by firms in $\mathscr{N}_r$ from suppliers outside r.

$GDP_r \equiv E_r^{dom} + E_r^{for} + Exp_r - Imp_r = \sum\limits_{i\in\mathscr{N}_r}(S_i - \sum\limits_{j\in\mathscr{N}_r} p_i\,x_{ji}) - \sum\limits_{j\notin\mathscr{N}_r} p_j \sum\limits_{i\in\mathscr{N}_r} x_{ij} = \sum\limits_{i\in\mathscr{N}_r}(S_i - \sum\limits_{j\in\mathscr{N}} p_j\,x_{ij})$. Using (10), and (12) $GDP_r = \sum_{i\in\mathscr{N}_r}(1 - \sum_{j\in\mathscr{N}}\Omega_{ij}^x)S_i = \sum_{i\in\mathscr{N}_r}(1 - \mu_i\,\omega_i^x)S_i$. This coincides with the total value-added generated firms in $\mathscr{N}_r$:

$$\begin{aligned} GDP_r &= \sum_{i\in\mathscr{N}_r}\big((\omega_i^\ell + \omega_i^x)S_i - \sum_{j\in\mathscr{N}}\Omega_{ij}^x S_i\big) \\ &= \sum_{i\in\mathscr{N}_r}\Big(\sum_{h\in\mathscr{H}} w_h\,\ell_{ih} - \mu_i\frac{\sum_{h\in\mathscr{H}} w_h\,\ell_{ih}}{\mu_i\,S_i}S_i + (1 - \mu_i\sum_{j\in\mathscr{N}}\widetilde{\Omega}_{ij}^x)S_i\Big) \\ &= \sum_{i\in\mathscr{N}_r}\Big(\sum_{h\in\mathscr{H}} w_h\,\ell_{ih} + (1-\mu_i)S_i\Big). \end{aligned} \tag{18}$$

In matrix form, GDPs are $\overline{GDP} = F_{\mathscr{N}}'\,(S - diag(S)\,\Omega_x\,\mathbb{1}_N)$, where $F_{\mathscr{N}}$ is a $N\times R$ selection matrix with $[F_{\mathscr{N}}] = 1$ if $i \in \mathscr{N}_r$ and 0 otherwise. Aggregate GDP is $GDP = \sum_{i\in\mathscr{N}}\left(1 - \sum_{j\in\mathscr{N}}\Omega_{ij}^x\right)S_i = \sum_{i\in\mathscr{N}}\left(1 - \mu_i\omega_i^x\right)S_i$

1.5 Proof for Proposition 1

Prices are given by $p_i^\ell = \frac{\sum_{h\in\mathscr{H}} w_h\,\ell_{ih}}{A_i^\ell\,Q_i^\ell(\{A_{ih}^\ell\,\ell_{ih}\}_{h\in\mathscr{H}})}$, $p_i^x = \frac{\sum_{j\in\mathscr{N}} p_j\,x_{ij}}{A_i^x\,Q_i^x(\{A_{ij}^x\,x_{ij}\}_{j\in\mathscr{N}})}$, $p_i = \frac{(p_i^\ell\,L_i + p_i^x\,X_i)}{\mu_i\,A_i\,Q_i(L_i,X_i)}$, and $p_h^c = \frac{\sum_{i\in\mathscr{N}} p_i\,C_{hi}}{Q_h^c(\{C_{hi}\}_{i\in\mathscr{N}})}$. Then $\widehat{p}_i^\ell = \overline{\frac{A_i^\ell}{p_i^\ell}\frac{\partial\,p_i^\ell}{\partial\,A_i^\ell}}\widehat{A}_i^\ell + \sum_{h\in\mathscr{H}}\big(\overline{\frac{w_h}{p_i^\ell}\frac{\partial\,p_i^\ell}{\partial\,w_h}}\widehat{w}_h + \overline{\frac{A_{ih}^\ell}{p_i^\ell}\frac{\partial\,p_i^\ell}{\partial\,A_{ih}^\ell}}\widehat{A}_{ih}^\ell + \overline{\frac{\ell_{ih}}{p_{z_i}^\ell}\frac{\partial\,p_i^\ell}{\partial\,\ell_{ih}}}\widehat{\ell}_{ih}\big)$, where the log deviation of any variable x around its steady-state value is $\widehat{x} = \log(x/\overline{x})$. Using $\overline{\frac{A_i^\ell}{p_i^\ell}\frac{\partial\,p_i^\ell}{\partial\,A_i^\ell}} = -1$, $\overline{\frac{w_h}{p_i^\ell}\frac{\partial\,p_i^\ell}{\partial\,w_h}} = \alpha_{ih}$, $\overline{\frac{A_{ih}^\ell}{p_i^\ell}\frac{\partial\,p_i^\ell}{\partial\,A_{ih}^\ell}} = -\alpha_{ih}$, and $\overline{\frac{\ell_{ih}}{p_i^\ell}\frac{\partial\,p_i^\ell}{\partial\,\ell_{ih}}} = \alpha_{ih} - e(L_i,\ell_{ih}) = 0$, we have that $\widehat{p}_i^\ell = -\widehat{A}_i^\ell + \sum_{h\in\mathscr{H}}\alpha_{ih}(\widehat{w}_h - \widehat{A}_{ih}^\ell)$, $\widehat{p}_i^x = -\widehat{A}_i^x + \sum_{j\in\mathscr{N}}\omega_{ij}(\widehat{p}_j - \widehat{A}_{ij}^x)$, $\widehat{p}_i = \omega_i^\ell\,\widehat{p}_i^\ell + \omega_i^x\,\widehat{p}_i^x - \widehat{A}_i - \widehat{\mu}_i$, and $\widehat{p}_h^c = \sum_{i\in\mathscr{N}}\beta_{hi}\,\widehat{p}_i$. In matrix form, these relationships are written as $\widehat{p}_\ell = \alpha\,\widehat{w} - \widehat{A}_\ell - (\alpha\circ\underline{\widehat{\mathrm{A}}}_\ell)\mathbb{1}_H$, $\widehat{p}_x = \mathscr{W}\,\widehat{p} - \widehat{A}_x - (\mathscr{W}\circ\underline{\widehat{\mathrm{A}}}_x)\mathbb{1}_N$, $\widehat{p} = diag(\omega_\ell)\widehat{p}_\ell + diag(\omega_x)\widehat{p}_x - \widehat{A} - \widehat{\mu}$, and $\widehat{p}_c = \beta\,\widehat{p}$. Then $\widehat{p} = \widetilde{\Psi}_x(\widetilde{\Omega}_\ell\,\widehat{w} - \widehat{\mathcal{A}} - \widehat{\mu})$ and $\widehat{p}_c = \widetilde{\mathscr{B}}(\widetilde{\Omega}_\ell\,\widehat{w} - \widehat{\mathcal{A}} - \widehat{\mu})$. The matrices previously used are: $\widehat{\mathcal{A}} \equiv \widehat{A} + diag(\omega_\ell)\widehat{A}_\ell + (\widetilde{\Omega}_\ell\circ\underline{\widehat{\mathrm{A}}}_\ell)\mathbb{1}_H + diag(\omega_x)\widehat{A}_x + (\widetilde{\Omega}_x\circ\underline{\widehat{\mathrm{A}}}_x)\mathbb{1}_N$, $\widehat{A} \equiv [\widehat{A}_1,\cdots,\widehat{A}_N']$, $\widehat{A}_\ell \equiv [\widehat{A}_1^\ell,\cdots,\widehat{A}_N^\ell]'$, $\widehat{A}_x \equiv [\widehat{A}_1^x,\cdots,\widehat{A}_N^x]'$, $\underline{\widehat{\mathrm{A}}}_\ell = [\underline{\widehat{\mathrm{A}}}_1^\ell,\cdots,\underline{\widehat{\mathrm{A}}}_N^\ell]'$, $\underline{\widehat{\mathrm{A}}}_i^\ell = [\widehat{A}_{i1}^\ell,\cdots,\widehat{A}_{iH}^\ell]'$, $\underline{\widehat{\mathrm{A}}}_x = [\underline{\widehat{\mathrm{A}}}_1^x,\cdots,\underline{\widehat{\mathrm{A}}}_n^x]'$, $\underline{\widehat{\mathrm{A}}}_i^x = [\widehat{A}_{i1}^x,\cdots,\widehat{A}_{iN}^x]'$, $\widehat{p} \equiv [\widehat{p}_1,\cdots,\widehat{p}_N]'$, $\widehat{p}_\ell \equiv [\widehat{p}_1^\ell,\cdots,\widehat{p}_N^\ell]'$, $\widehat{p}_x \equiv [\widehat{p}_1^x,\cdots,\widehat{p}_N^x]'$, $\widehat{\mu} \equiv [\widehat{\mu}_1,\cdots,\widehat{\mu}_N]'$, and $\widehat{w} \equiv [\widehat{w}_1,\cdots,\widehat{w}_H]'$.

1.6 Proof for Theorem 1

From (1), the first-order approximation to cluster-r value added is

$$GDP_r\widehat{GDP}_r = \sum_{i\in\mathscr{N}_r} S_i(\widehat{S}_i - \sum_{j\in\mathscr{N}}\mu_i\,\omega_i^x\,\omega_{ij}(\widehat{\mu}_i + \widehat{\omega}_i^x + \widehat{\omega}_{ij} + \widehat{S}_i)).$$

From (12) and the identity $\widehat{p}_x = \mathscr{W}\,\widehat{p} - \widehat{A}_x - (\mathscr{W}\circ\underline{\widehat{\mathrm{A}}}_x)\mathbb{1}_N$,

$$\Phi_r\,\widehat{GDP}_r = \sum_{i\in\mathscr{N}_r}\lambda_i(\widehat{S}_i - \sum_{j\in\mathscr{N}}\Omega_{ij}^x(\widehat{p}_j + \widehat{x}_{ij})). \tag{19}$$

From here, the Divisia (first-order) GDP deflator for cluster r

$$\widehat{p}_{Y_r} = \sum_{i\in\mathscr{N}_r}\frac{\lambda_i}{\Phi_r}(\widehat{p}_i - \sum_{j\in\mathscr{N}}\Omega_{ij}^x\,\widehat{p}_j). \tag{20}$$

Hence, first-order real value-added growth for cluster r is

$$\Phi_r\,\widehat{Y}_r = \sum_{i\in\mathscr{N}_r}\lambda_i(\widehat{y}_i - \sum_{j\in\mathscr{N}}\Omega_{ij}^x\,\widehat{x}_{ij}). \tag{21}$$

Using the first-order approximation to goods-market clearing, (21) can be written as

$$\Phi_r\,\widehat{Y}_r = \sum_{i\in\mathscr{N}_r}(\sum_{h\in\mathscr{H}}\beta_{hi}\,\chi_h\,\widehat{C}_{hi} + \sum_{j\in\mathscr{N}}(\Omega_{ji}^x\,\lambda_j\,\widehat{x}_{ji} - \Omega_{ij}^x\,\lambda_i\,\widehat{x}_{ij})).$$

Using $\widehat{p} = \widetilde{\Psi}_x(\widetilde{\Omega}_\ell\, \widehat{w} - \widehat{\mathcal{A}} - \widehat{\mu})$ and first-order approximations for (12), (14), and (17)

$$
\begin{aligned}
\Phi_r\, \widehat{Y}_r = & \sum_{h\in\mathcal{H}} \chi_h \sum_{i\in\mathcal{N}_r} \beta_{hi}\, \widehat{\beta}_{hi} + \sum_{h\in\mathcal{H}} \chi_h (\sum_{i\in\mathcal{N}_r} \beta_{hi}) \widehat{E}_h \\
& + \sum_{j\in\mathcal{N}} \mu_j\, \omega_j^x\, \lambda_j \sum_{i\in\mathcal{N}_r} \omega_{ji}\, \widehat{\omega}_{ji} - \sum_{i\in\mathcal{N}_r} \mu_i\, \omega_i^x\, \lambda_i \sum_{j\in\mathcal{N}} \omega_{ij}\, \widehat{\omega}_{ij} \\
& + \sum_{i\in\mathcal{N}_r} \sum_{j\in\mathcal{N}} \Omega_{ji}^x\, \lambda_j (\widehat{\omega}_j^x + \widehat{\mu}_j) - \sum_{i\in\mathcal{N}_r} \sum_{j\in\mathcal{N}} \Omega_{ij}^x\, \lambda_i (\widehat{\omega}_i^x + \widehat{\mu}_i) \\
& + \sum_{i\in\mathcal{N}_r} \sum_{j\in\mathcal{N}} \Omega_{ji}^x\, \lambda_j\, \widehat{S}_j - \sum_{i\in\mathcal{N}_r} \lambda_i (\sum_{j\in\mathcal{N}} \Omega_{ij}^x) \widehat{S}_i \\
& + \sum_{i\in\mathcal{N}_r} \mu_i\, \omega_i^x\, \lambda_i \sum_{q\in\mathcal{R}} \sum_{j\in\mathcal{N}_q} \omega_{ij}\, \widehat{p}_j - \sum_{i\in\mathcal{N}_r} \underbrace{(\sum_{h\in\mathcal{H}} \beta_{hi}\, \chi_h + \sum_{j\in\mathcal{N}} \Omega_{ji}^x\, \lambda_j)}_{=\lambda_i} \widehat{p}_i.
\end{aligned}
$$

$$
\begin{aligned}
\Phi_r\, \widehat{Y}_r = & \sum_{h\in\mathcal{H}} \chi_h \sum_{i\in\mathcal{N}_r} \beta_{hi}\, \widehat{\beta}_{hi} + \sum_{h\in\mathcal{H}} (\sum_{j\in\mathcal{N}_r} \beta_{hj}) (\Lambda_h (\widehat{w}_h + \widehat{L}_h) + \tau_h\, \widehat{T}_h) \\
& + \sum_{h\in\mathcal{H}} (\sum_{j\in\mathcal{N}_r} \beta_{hj}) \sum_{i\in\mathcal{N}} \kappa_{ih}\, \lambda_i ((1-\mu_i)(\widehat{\kappa}_{ih} + \widehat{S}_i) - \mu_i\, \widehat{\mu}_i) + \sum_{j\in\mathcal{N}} \mu_j\, \omega_j^x\, \lambda_j \sum_{i\in\mathcal{N}_r} \omega_{ji}\, \widehat{\omega}_{ji} \\
& - \sum_{i\in\mathcal{N}_r} \mu_i\, \omega_i^x\, \lambda_i \sum_{j\in\mathcal{N}} \omega_{ij}\, \widehat{\omega}_{ij} + \sum_{j\in\mathcal{N}} \mu_j\, \omega_j^x\, \lambda_j (\sum_{i\in\mathcal{N}_r} \omega_{ji}) (\widehat{\omega}_j^x + \widehat{\mu}_j) \\
& - \sum_{i\in\mathcal{N}_r} \mu_i\, \omega_i^x\, \lambda_i (\sum_{j\in\mathcal{N}} \omega_{ij}) (\widehat{\omega}_i^x + \widehat{\mu}_i) + \sum_{i\in\mathcal{N}_r} \sum_{j\in\mathcal{N}} \Omega_{ji}^x\, \lambda_j\, \widehat{S}_j - \sum_{i\in\mathcal{N}_r} \lambda_i (\sum_{j\in\mathcal{N}} \Omega_{ij}^x) \widehat{S}_i \\
& + \sum_{j\in\mathcal{N}} (\sum_{i\in\mathcal{N}_r} \lambda_i (\widetilde{\psi}_{ij}^x - \sum_{m\in\mathcal{N}} \Omega_{im}^x\, \widetilde{\psi}_{mj}^x)) (\widehat{\mathcal{A}}_j + \widehat{\mu}_j) \\
& - \sum_{h\in\mathcal{H}} (\sum_{i\in\mathcal{N}_r} \lambda_i (\widetilde{\psi}_{ih}^\ell - \sum_{m\in\mathcal{N}} \Omega_{im}^x\, \widetilde{\psi}_{mh}^\ell)) (\widehat{w}_h + \widehat{L}_h) \\
& + \sum_{h\in\mathcal{H}} (\sum_{i\in\mathcal{N}_r} \lambda_i (\widetilde{\psi}_{ih}^\ell - \sum_{m\in\mathcal{N}} \Omega_{im}^x\, \widetilde{\psi}_{mh}^\ell)) \widehat{L}_h.
\end{aligned}
$$

Before continuing, define the following objects

- Given $\widetilde{\Psi}_x = \sum_{q=0}^{\infty} \widetilde{\Omega}_x^q = I_N + \widetilde{\Omega}_x \widetilde{\Psi}_x$, we have $\widetilde{\Psi}_x - \Omega_x \widetilde{\Psi}_x = I_N + (\widetilde{\Omega}_x - \Omega_x)\widetilde{\Psi}_x$. In the absence of distortions ($\widetilde{\Omega}_x = \Omega_x$), $\widetilde{\psi}_{ij}^x - \sum_{m\in\mathcal{N}} \Omega_{im}^x \widetilde{\psi}_{mj}^x = \mathbb{1}\{i=j\}$, so exposure is purely direct. Consequently, a shock to firm i affects Y_r only of $i \in \mathcal{N}_r$, implying no cross-cluster spillovers.

  With distortions, $\widetilde{\psi}_{ij}^x - \sum_{m\in\mathcal{N}} \Omega_{im}^x \widetilde{\psi}_{mj}^x = \mathbb{1}\{i=j\} + \sum_{m\in\mathcal{N}} (\widetilde{\Omega}_{im}^x - \Omega_{im}^x) \widetilde{\psi}_{mj}^x$, so firm i's exposure to shock in j has a direct component $\mathbb{1}\{i=j\}$, and an indirect component that aggregates higher-order supply chain effects. The latter is weighted by the cost-revenue wedge $\widetilde{\Omega}_{im}^x - \Omega_{im}^x$ along each supplier m. Accordingly, define the rent-adjusted downstream exposure $\widetilde{\psi}_{ij,\Delta\Omega}^x = \sum_{m\in\mathcal{N}} (\widetilde{\Omega}_{im}^x - \Omega_{im}^x) \widetilde{\psi}_{mj}^x$, which summarizes the extend to which changes in j's input prices transmit (directly or indirectly) into firm i's costs once rents are accounted for. With distortions, a shock to firm j has a first-order effect on Y_r proportional to $\mathbb{1}\{i=j\}+\widetilde{\psi}^x_{ij,\Delta\Omega}$. Hence, favorable shocks can raise real value added in cluster r either because $j \in \mathscr{N}_r$ or because firms in $\mathscr{N}_r$ are downstream-exposed to j; the latter channel reflects rents appropriated by downstream firms in cluster r.

For this reason, $\ddot{\lambda}^r_j = \mathbb{1}\{j \in \mathscr{N}_r\}\frac{\lambda_j}{\Phi_r} + \sum_{i \in \mathscr{N}_r} \frac{\lambda_i}{\Phi_r}\widetilde{\psi}^x_{ij,\Delta\Omega}$ is the rent-adjusted share of cluster r's value added that passes through firm j. Value added flows through j via two channels: (i) directly, when $j \in \mathscr{N}_r$ sells its own output; and (ii) indirectly, when firms $i \in \mathscr{N}_r$ use inputs sourced from j (directly or indirectly) to produce goods sold at a markup, generating rents proportional $1-\mu_i$. Both channels are Domar-weighted and normalized by Φ_r. Without intermediate inputs, $\sum_{i\in\mathscr{N}_r}\ddot{\lambda}^r_i = 1$. With intermediate inputs and no distortions r ($\mu_i = 1$ for all $i \in \mathscr{N}_r$), $\sum_{i\in\mathscr{N}_r}\ddot{\lambda}^r_i = \sum_{i\in\mathscr{N}_r}\frac{\lambda_i}{\Phi_r} \geq 1$. In general, $\sum_{i\in\mathscr{N}_r}\ddot{\lambda}^r_i \geq 1$.

For the aggregate economy, $\ddot{\lambda}^G_j = \lambda_j + \sum_{i\in\mathscr{N}}\lambda_i\sum_{m\in\mathscr{N}}(\widetilde{\Omega}^x_{im}-\Omega^x_{im})\widetilde{\psi}^x_{mj}$. In vector form, $\ddot{\lambda}^G = (I_N + \widetilde{\Psi}'_x(\widetilde{\Omega}_x-\Omega_x)')\lambda = \widetilde{\Psi}'_x(I_N-\Omega'_x)\lambda$. Moreover, using $\lambda = \Psi'_x\,\beta\,\chi = \Psi'_x\left(I_N-\widetilde{\Omega}'_x\right)\widetilde{\Psi}'_x\,\beta\,\chi = \Psi'_x\left(I_N-\widetilde{\Omega}'_x\right)\widetilde{\lambda}$, it follows that $\ddot{\lambda}^G = \widetilde{\lambda}$.

- Similarly, since $\widetilde{\Psi}_\ell = \widetilde{\Psi}_x\widetilde{\Omega}_\ell$ and $\widetilde{\Psi}_x = I_N + \widetilde{\Omega}_x\widetilde{\Psi}_x$, we have $\widetilde{\Psi}_\ell - \Omega_x\widetilde{\Psi}_\ell = \widetilde{\Omega}_\ell + (\widetilde{\Omega}_x-\Omega_x)\widetilde{\Psi}_\ell$. When $\Omega_x = \widetilde{\Omega}_x$ (no distortions), $\widetilde{\psi}^\ell_{ih} - \sum_{j\in\mathscr{N}}\Omega^x_{ij}\widetilde{\psi}^\ell_{jh} = \widetilde{\Omega}^\ell_{ih}$, so firm i's exposure to factor h's cost is purely direct. With distortions ($\Omega_x \neq \widetilde{\Omega}_x$), $\widetilde{\Omega}^\ell_{ih} + \sum_{j\in\mathscr{N}}(\widetilde{\Omega}^x_{ij}-\Omega^x_{ij})\widetilde{\psi}^\ell_{jh}$, which combines the direct exposure $\widetilde{\Omega}^\ell_{ih}$ with higher-round network effects. Accordingly, define the rent-adjusted downstream exposure of firm i to factor h as $\widetilde{\psi}^\ell_{ih,\Delta\Omega} \equiv \sum_{j\in\mathscr{N}}(\widetilde{\Omega}^x_{ij}-\Omega^x_{ij})\widetilde{\psi}^\ell_{jh}$, which aggregates the paths through which factor h's cost loads into firm i's unit cost, weighted by the cost-revenue wedge $\widetilde{\Omega}^x_{ij}-\Omega^x_{ij}$ along each link.

  For this reason, $\ddot{\Lambda}^r_h = \sum_{i\in\mathscr{N}_r}\frac{\lambda_i}{\Phi_r}(\widetilde{\Omega}^\ell_{ih} + \widetilde{\psi}^\ell_{ih,\Delta\Omega})$ is the share of cluster-r value added that can be traced back to factor h. The collection $\{\ddot{\Lambda}^r_h\}_{h\in\mathscr{H}}$ forms a probability distribution because

$$\sum_{h\in\mathscr{H}} \ddot{\Lambda}^r_h = \sum_{i\in\mathscr{N}_r} \frac{\lambda_i}{\Phi_r} \underbrace{\sum_{h\in\mathscr{H}} \widetilde{\Omega}^\ell_{ih}}_{=\omega^\ell_i} + \sum_{i\in\mathscr{N}_r} \frac{\lambda_i}{\Phi_r} \sum_{j\in\mathscr{N}} (\widetilde{\Omega}^x_{ij} - \Omega^x_{ij}) \underbrace{\sum_{h\in\mathscr{H}} \widetilde{\psi}^\ell_{jh}}_{=1}$$

$$= \sum_{i\in\mathscr{N}_r} \omega^\ell_i \frac{\lambda_i}{\Phi_r} + \sum_{i\in\mathscr{N}_r} \frac{\lambda_i}{\Phi_r} (\underbrace{\sum_{j\in\mathscr{N}} \widetilde{\Omega}^x_{ij}}_{=\omega^x_i} - \sum_{j\in\mathscr{N}} \Omega^x_{ij}) = \sum_{i\in\mathscr{N}_r} (1 - \sum_{j\in\mathscr{N}} \Omega^x_{ij}) \frac{\lambda_i}{\Phi_r} = 1,$$

where the last equality follows from (1). Without distortions in r, $\ddot{\Lambda}^r_h = \frac{1}{\Phi_r} \sum_{i\in\mathscr{N}_r} \Omega^\ell_{ih} \lambda_i$; the same reduction applies when factors are cluster-specific. For the aggregate economy, $\ddot{\Lambda}^G_h = \sum_{i\in\mathscr{H}} \lambda_i (\widetilde{\Omega}^\ell_{ih} + \sum_{j\in\mathscr{N}} (\widetilde{\Omega}^x_{ij} - \Omega^x_{ij}) \widetilde{\psi}^\ell_{jh})$, and in vector form $\ddot{\Lambda}^G = \widetilde{\Psi}'_\ell (I_N - \Omega'_x)\lambda = \widetilde{\Lambda}$.

- $\beta_{h|r} \equiv \sum_{i\in\mathscr{N}_r} \beta_{hi}$ is household h's direct expenditure intensity on cluster r.
- $\Omega^x_{j|r} \equiv \sum_{i\in\mathscr{N}_r} \Omega^x_{ji}$ is firm j's direct intermediate-input intensity on cluster r.

Therefore,

$$\begin{aligned}
\Phi_r \widehat{Y}_r = &\sum_{h\in\mathscr{H}} \chi_h \sum_{i\in\mathscr{N}_r} \beta_{hi} \widehat{\beta}_{hi} + \sum_{h\in\mathscr{H}} \beta_{h|r} (\Lambda_h \widehat{J}_h + \tau_h \widehat{T}_h) \\
&+ \sum_{h\in\mathscr{H}} \beta_{h|r} \sum_{i\in\mathscr{N}} \kappa_{ih} \lambda_i \left((1-\mu_i) \left(\widehat{\kappa}_{ih} + \widehat{S}_i \right) - \mu_i \widehat{\mu}_i \right) - \Phi_r \sum_{h\in\mathscr{H}} \ddot{\Lambda}^r_h \widehat{J}_h \\
&+ \sum_{i\in\mathscr{N}} \Omega^x_{i|r} \lambda_i (\widehat{\omega}^x_i + \widehat{\mu}_i) - \sum_{q\in\mathscr{R}} \sum_{i\in\mathscr{N}_r} \Omega^x_{i|q} \lambda_i (\widehat{\omega}^x_i + \widehat{\mu}_i) \\
&+ \sum_{i\notin\mathscr{N}_r} \lambda_i \sum_{j\in\mathscr{N}_r} \Omega^x_{ij} \widehat{\omega}_{ij} - \sum_{i\in\mathscr{N}_r} \lambda_i \sum_{j\notin\mathscr{N}_r} \Omega^x_{ij} \widehat{\omega}_{ij} + \Phi_r \sum_{j\in\mathscr{N}} \ddot{\lambda}^r_j \left(\widehat{\mathcal{A}}_j + \widehat{\mu}_j \right) \\
&+ \sum_{i\in\mathscr{N}_r} \sum_{j\in\mathscr{N}} \Omega^x_{ji} \lambda_j \widehat{S}_j - \sum_{q\in\mathscr{R}} \sum_{i\in\mathscr{N}_r} \Omega^x_{i|q} \lambda_i \widehat{S}_i + \Phi_r \sum_{h\in\mathscr{H}} \ddot{\Lambda}^r_h \widehat{L}_h.
\end{aligned}$$

Consequently, there exists a CRS aggregator $F_r(\{L_h\}_{h\in\mathscr{H}})$ such that $Y_r = TFP_r\, F_r(\{L_h\}_{h\in\mathscr{H}})$, where $F_r(\{L_h\}_{h\in\mathscr{H}})$, and, locally, $\frac{d \log F_r(\{L_b\}_{b\in\mathscr{H}})}{d \log L_h} = \ddot{\Lambda}^r_h$.

Add and subtract $\widehat{GDP}$ to express the previous equation as

$$\begin{aligned}
\Phi_r\,\widehat{Y}_r &= \sum_{h\in\mathcal{H}}\chi_h\sum_{i\in\mathcal{N}_r}\beta_{hi}\,\widehat{\beta}_{hi} + \sum_{h\in\mathcal{H}}\beta_{h|r}(\Lambda_h\,\widehat{\Lambda}_h + \tau_h\,\widehat{\tau}_h) - \Phi_r\sum_{h\in\mathcal{H}}\ddot{\Lambda}^r_h\,\widehat{\Lambda}_h \\
&+ \sum_{h\in\mathcal{H}}\beta_{h|r}\sum_{i\in\mathcal{N}}\kappa_{ih}\,\lambda_i((1-\mu_i)(\widehat{\kappa}_{ih}+\widehat{\lambda}_i) - \mu_i\,\widehat{\mu}_i) + \sum_{i\in\mathcal{N}}\Omega^x_{i|r}\,\lambda_i(\widehat{\omega}^x_i + \widehat{\mu}_i + \widehat{\lambda}_i) \\
&- \sum_{q\in\mathcal{R}}\sum_{i\in\mathcal{N}_r}\Omega^x_{i|q}\,\lambda_i(\widehat{\omega}^x_i + \widehat{\mu}_i + \widehat{\lambda}_i) + \sum_{i\notin\mathcal{N}_r}\lambda_i\sum_{j\in\mathcal{N}_r}\Omega^x_{ij}\,\widehat{\omega}_{ij} - \sum_{i\in\mathcal{N}_r}\lambda_i\sum_{j\notin\mathcal{N}_r}\Omega^x_{ij}\,\widehat{\omega}_{ij} \\
&+ \Phi_r\sum_{j\in\mathcal{N}}\ddot{\lambda}^r_j(\widehat{\mathcal{A}}_j + \widehat{\mu}_j) + \Big(\sum_{h\in\mathcal{H}}\beta_{h|r}(\Lambda_h + \sum_{i\in\mathcal{N}_q}\kappa_{ih}(1-\mu_i)\lambda_i + \tau_h) \\
&+ \sum_{i\in\mathcal{N}}\Omega^x_{j|r}\,\lambda_j - \sum_{q\in\mathcal{R}}\sum_{i\in\mathcal{N}_r}\Omega^x_{i|q}\,\lambda_i - \Phi_r\Big)\widehat{GDP} + \Phi_r\sum_{h\in\mathcal{H}}\ddot{\Lambda}^r_h\,\widehat{L}_h.
\end{aligned}$$

From (1), (17), and $d\,\chi_h = d\,\Lambda_h + \sum_{i\in\mathcal{N}}\kappa_{ih}\,\lambda_i\,((1-\mu_i)\,d\,\log\,\kappa_{ih}\,\lambda_i - d\,\mu_i) + d\,\tau_h$

$$\begin{aligned}
\Phi_r\,\widehat{Y}_r &= \sum_{h\in\mathcal{H}}\chi_h\sum_{i\in\mathcal{N}_r}\beta_{hi}\,\widehat{\beta}_{hi} + \sum_{h\in\mathcal{H}}(\beta_{h|r}\,\chi_h\,\widehat{\chi}_h - \Phi_r\,\ddot{\Lambda}^r_h\,\widehat{\Lambda}_h) + \Phi_r\sum_{j\in\mathcal{N}}\ddot{\lambda}^r_j(\widehat{\mathcal{A}}_j + \widehat{\mu}_j) \\
&+ \sum_{i\in\mathcal{N}}\Omega^x_{i|r}\,\lambda_i(\widehat{\omega}^x_i + \widehat{\mu}_i + \widehat{\lambda}_i) - \sum_{q\in\mathcal{R}}\sum_{i\in\mathcal{N}_r}\Omega^x_{i|q}\,\lambda_i(\widehat{\omega}^x_i + \widehat{\mu}_i + \widehat{\lambda}_i) \\
&+ \sum_{i\notin\mathcal{N}_r}\lambda_i\sum_{j\in\mathcal{N}_r}\Omega^x_{ij}\,\widehat{\omega}_{ij} - \sum_{i\in\mathcal{N}_r}\lambda_i\sum_{j\notin\mathcal{N}_r}\Omega^x_{ij}\,\widehat{\omega}_{ij} + \Phi_r\sum_{h\in\mathcal{H}}\ddot{\Lambda}^r_h\,\widehat{L}_h \\
&+ \underbrace{\Big(\sum_{h\in\mathcal{H}}\beta_{h|r}\,\chi_h + \sum_{i\in\mathcal{N}}\Omega^x_{j|r}\,\lambda_j - \sum_{q\in\mathcal{R}}\sum_{i\in\mathcal{N}_r}\Omega^x_{i|q}\,\lambda_i - \Phi_r\Big)}_{=0}\widehat{GDP}.
\end{aligned}$$

The distribution of revenue sources for cluster r is the $H + N + dim\,(\mathcal{N}_r)$ vector $\mathcal{R}'_r \equiv \Phi_r^{-1}\left[\left\{\beta_{h|r}\,\chi_h\right\}_{h\in\mathcal{H}}\quad\left\{\Omega^x_{i|r}\,\lambda_i\right\}_{i\in\mathcal{N}}\quad-\left\{\lambda_i\sum_{q\in\mathcal{R}}\Omega^x_{i|q}\right\}_{i\in\mathcal{N}_r}\right]$.
The elements of this vector sum to one, and a first-order change in Φ_r is

$$\begin{aligned}
d\,\Phi_r &= \sum_{h\in\mathcal{H}}\beta_{h|r}\,d\,\chi_h + \sum_{h\in\mathcal{H}}\chi_h\sum_{i\in\mathcal{N}_r}d\,\beta_{hi} + \sum_{j\notin\mathcal{N}_r}\mu_j\,\omega^x_j\,\lambda_j\sum_{i\in\mathcal{N}_r}d\,\omega_{ji} - \sum_{i\in\mathcal{N}_r}\mu_i\,\omega^x_i\,\lambda_i\sum_{j\notin\mathcal{N}_r}d\,\omega_{ij} \\
&+ \sum_{i\in\mathcal{N}}\Omega^x_{i|r}\,\lambda_i(\widehat{\omega}^x_i + \widehat{\mu}_i + \widehat{\lambda}_i) - \sum_{q\in\mathcal{R}}\sum_{i\in\mathcal{N}_r}\Omega^x_{i|q}\,\lambda_i(\widehat{\omega}^x_i + \widehat{\mu}_i + \widehat{\lambda}_i) = \sum_{h\in\mathcal{H}}\beta_{h|r}\,d\,\chi_h \\
&+ \sum_{h\in\mathcal{H}}\chi_h\,d\,\beta_{h|r} + \sum_{i\notin\mathcal{N}_r}\lambda_i\,d\,\Omega^x_{i|r} - \sum_{i\in\mathcal{N}_r}\lambda_i\sum_{j\notin\mathcal{N}_r}d\,\Omega^x_{ij} + \sum_{i\notin\mathcal{N}_r}\Omega^x_{i|r}\,d\,\lambda_i - \sum_{i\in\mathcal{N}_r}\Big(\sum_{j\notin\mathcal{N}_r}\Omega^x_{ij}\Big)d\,\lambda_i.
\end{aligned}$$

Using results from Theorem 1 in Rojas-Bernal (2026), the first-order variations in Λ and λ are given by $d\,\Lambda_h = \sum_{b\in\mathcal{H}}\mathscr{C}_{bh}\,d\,\chi_b + \sum_{i\in\mathcal{N}}\psi^\ell_{ih}\,\lambda_i\,d\,\log\,\mu_i + \sum_{i\in\mathcal{N}}\mu_i\,\lambda_i\,d\,\widetilde{\Omega}^\ell_{ih} + \sum_{j\in\mathcal{N}}\psi^\ell_{jh}\big(\sum_{b\in\mathcal{H}}\chi_b\,d\,\beta_{bj} + \sum_{i\in\mathcal{N}}\mu_i\,\lambda_i\,d\,\widetilde{\Omega}^x_{ij}\big)$, and $d\,\lambda_i = \sum_{h\in\mathcal{H}}\mathscr{B}_{hi}\,d\,\chi_h + \sum_{j\in\mathcal{N}}\psi^x_{ji}\,\lambda_j\,d\,\log\,\mu_j +$

$\sum\limits_{j\in\mathcal{N}} \psi^x_{ji}\big(\sum\limits_{h\in\mathcal{H}} \chi_h\, d\,\beta_{hj} + \sum\limits_{m\in\mathcal{N}} \mu_m\, \lambda_m\, d\,\widetilde{\Omega}^x_{mj}\big)$. Then

$$\begin{aligned} d\,\Phi_r = &\sum_{h\in\mathcal{H}} (\beta_{h|r} + \sum_{i\notin\mathcal{N}_r} \mathscr{B}_{hi}\,\Omega^x_{i|r} - \sum_{i\in\mathcal{N}_r} \mathscr{B}_{hi} \sum_{j\notin\mathcal{N}_r} \Omega^x_{ij}) d\,\chi_h + \sum_{h\in\mathcal{H}} \chi_h\, d\,\beta_{h|r} \\ &+ \sum_{i\in\mathcal{N}} \lambda_i (\sum_{j\notin\mathcal{N}_r} \psi^x_{ij}\,\Omega^x_{j|r} - \sum_{j\in\mathcal{N}_r} \psi^x_{ij} \sum_{m\notin\mathcal{N}_r} \Omega^x_{jm}) d\,\log\,\mu_j + \sum_{i\notin\mathcal{N}_r} \lambda_i\, d\,\Omega^x_{i|r} \\ &+ \sum_{h\in\mathcal{H}} \chi_h \sum_{i\in\mathcal{N}} (\sum_{j\notin\mathcal{N}_r} \psi^x_{ij}\,\Omega^x_{j|r} - \sum_{j\in\mathcal{N}_r} \psi^x_{ij} \sum_{m\notin\mathcal{N}_r} \Omega^x_{jm}) d\,\beta_{hi} - \sum_{i\notin\mathcal{N}_r} \sum_{j\in\mathcal{N}_r} \lambda_j\, d\,\Omega^x_{ji} \\ &+ \sum_{i\in\mathcal{N}} \mu_i\,\lambda_i \sum_{j\in\mathcal{N}} (\sum_{m\notin\mathcal{N}_r} \psi^x_{jm}\,\Omega^x_{m|r} - \sum_{m\in\mathcal{N}_r} \psi^x_{jm} \sum_{f\notin\mathcal{N}_r} \Omega^x_{mf}) d\,\widetilde{\Omega}^x_{ij}. \end{aligned}$$

This implies that

$$\widehat{Y}_r = \sum_{j\in\mathcal{N}} \ddot{\lambda}^r_j (\widehat{\mathcal{A}}_j + \widehat{\mu}_j) + \widehat{\Phi}_r - \sum_{h\in\mathcal{H}} \ddot{\Lambda}^r_h\, \widehat{\Lambda}_h + \sum_{h\in\mathcal{H}} \ddot{\Lambda}^r_h\, \widehat{L}_h. \tag{22}$$

Additionally, using $d\,\Lambda_h$, $\delta^r_h = \frac{\ddot{\Lambda}^r_h}{\Lambda_h}$, $M^r_h = \sum_{b\in\mathcal{H}} \mathscr{C}_{hb}\,\delta^r_b$, and $F^r_i = \sum_{h\in\mathcal{H}} \psi^\ell_{ih}\,\delta^r_h$

$$\begin{aligned} \sum_{h\in\mathcal{H}} \ddot{\Lambda}^r_h\, \widehat{\Lambda}_h = &\sum_{h\in\mathcal{H}} \delta^r_h\, d\,\Lambda_h = \sum_{h\in\mathcal{H}} M^r_h\, d\,\chi_h + \sum_{h\in\mathcal{H}} \chi_h \sum_{b\in\mathcal{H}} \delta^r_b\, d\,\mathscr{C}_{hb} \\ = &\sum_{h\in\mathcal{H}} M^r_h\, d\,\chi_h + \sum_{i\in\mathcal{N}} \lambda_i\, F^r_i\, d\,\log\,\mu_i + \sum_{i\in\mathcal{N}} \mu_i\,\lambda_i \sum_{h\in\mathcal{H}} \delta^r_h\, d\,\widetilde{\Omega}^\ell_{ih} \\ &+ \sum_{h\in\mathcal{H}} \chi_h \sum_{i\in\mathcal{N}} F^r_i\, d\,\beta_{hi} + \sum_{i\in\mathcal{N}} \mu_i\,\lambda_i \sum_{j\in\mathcal{N}} F^r_j\, d\,\widetilde{\Omega}^x_{ij}. \end{aligned}$$

Then

$$\begin{aligned} \widehat{TFP}_r = &\sum_{i\in\mathcal{N}} \ddot{\lambda}^r_i \left(\widehat{\mathcal{A}}_i + \widehat{\mu}_i\right) + \widehat{\Phi}_r - \sum_{h\in\mathcal{H}} M^r_h\, d\,\chi_h - \sum_{h\in\mathcal{H}} \chi_h \sum_{b\in\mathcal{H}} \delta^r_b\, d\,\mathscr{C}_{hb} \\ = &\sum_{i\in\mathcal{N}} \ddot{\lambda}^r_i \left(\widehat{\mathcal{A}}_i + \widehat{\mu}_i\right) + \widehat{\Phi}_r - \sum_{h\in\mathcal{H}} M^r_h\, d\,\chi_h - \sum_{i\in\mathcal{N}} \lambda_i\, F^r_i\, d\,\log\,\mu_i \\ &- \sum_{i\in\mathcal{N}} \mu_i\,\lambda_i \sum_{h\in\mathcal{H}} \delta^r_h\, d\,\widetilde{\Omega}^\ell_{ih} - \sum_{h\in\mathcal{H}} \chi_h \sum_{i\in\mathcal{N}} F^r_i\, d\,\beta_{hi} - \sum_{i\in\mathcal{N}} \mu_i\,\lambda_i \sum_{j\in\mathcal{N}} F^r_j\, d\,\widetilde{\Omega}^x_{ij}. \end{aligned} \tag{23}$$

1.6.1 Proof for Corollary 1

(a) Taking χ, β, $\widetilde{\Omega}_\ell$, and $\widetilde{\Omega}_x$ as given: $\frac{\partial\,\log\,TFP_r}{\partial\,\log\,\mu_i} = \ddot{\lambda}^r_i + \frac{\lambda_i}{\Phi_r}\psi_{i|r} + \frac{1}{\Phi_r}\big(\sum_{i\notin\mathcal{N}_r} \lambda_i \frac{\partial\,\Omega^x_{i|r}}{\partial\,\log\,\mu_i} - \sum_{i\notin\mathcal{N}_r}\sum_{j\in\mathcal{N}_r} \lambda_j \frac{\partial\,\Omega^x_{ji}}{\partial\,\log\,\mu_i}\big) - \lambda_i\,F^r_i = \ddot{\lambda}^r_i + \frac{\lambda_i}{\Phi_r}(\psi_{i|r} + \mu_i(\mathbb{1}_{\{i\notin\mathcal{N}_r\}}\sum_{j\in\mathcal{N}_r}\widetilde{\Omega}^x_{ij} - \mathbb{1}_{\{i\in\mathcal{N}_r\}}\sum_{j\notin\mathcal{N}_r}\widetilde{\Omega}^x_{ij})) - \lambda_i\,F^r_i$. (b) Taking λ, β, $\widetilde{\Omega}_\ell$, and $\widetilde{\Omega}_x$ as given: $\frac{\partial\,\log\,TFP_r}{\partial\,\Lambda_h} - \frac{\partial\,\log\,TFP_r}{\partial\,\Lambda_b} = \frac{1}{\Phi_r}(\beta_{h|r} - \beta_{b|r} + \mathscr{B}_{h|r} - \mathscr{B}_{b|r}) + \delta^r_b - \delta^r_h$. (c) Taking β, $\widetilde{\Omega}_\ell$, and $\widetilde{\Omega}_x$ as given: $\frac{\partial\,\log\,TFP_r}{\partial\,\chi_h} - \frac{\partial\,\log\,TFP_r}{\partial\,\chi_b} = \frac{1}{\Phi_r}(\beta_{h|r} - \beta_{b|r} + \mathscr{B}_{h|r} - \mathscr{B}_{b|r}) + M^r_b - M^r_h$. (d) Taking χ, β, and $\widetilde{\Omega}_x$ as given: $\frac{\partial\,\log\,TFP_r}{\partial\,\widetilde{\Omega}^\ell_{ih}} - \frac{\partial\,\log\,TFP_r}{\partial\,\widetilde{\Omega}^\ell_{ib}} = \mu_i\,\lambda_i(\delta^r_b - \delta^r_h)$. (e) Taking χ, $\widetilde{\Omega}_\ell$, and $\widetilde{\Omega}_x$ as given: $\frac{\partial\,\log\,TFP_r}{\partial\,\beta_{hi}} - \frac{\partial\,\log\,TFP_r}{\partial\,\beta_{hj}} = \frac{\partial\,\log\,\Phi_r}{\partial\,\beta_{hi}} - \frac{\partial\,\log\,\Phi_r}{\partial\,\beta_{hj}} - \chi_h(F^r_i - F^r_j) = \frac{\chi_h}{\Phi_r}(\psi_{i|r} - \psi_{j|r} + \frac{\partial\,\beta_{h|r}}{\partial\,\beta_{hi}} - \frac{\partial\,\beta_{h|r}}{\partial\,\beta_{hj}}) - \chi_h(F^r_i - F^r_j)$, where $\frac{\partial\,\beta_{h|r}}{\partial\,\beta_{hi}} - \frac{\partial\,\beta_{h|r}}{\partial\,\beta_{hj}} = \mathbb{1}_{\{i\in\mathcal{N}_r, j\notin\mathcal{N}_r\}} - \mathbb{1}_{\{j\in\mathcal{N}_r, i\notin\mathcal{N}_r\}} = \mathbb{1}_{\{i\in\mathcal{N}_r\}}(1 - \mathbb{1}_{\{j\in\mathcal{N}_r\}}) - \mathbb{1}_{\{j\in\mathcal{N}_r\}}(1 - \mathbb{1}_{\{i\in\mathcal{N}_r\}}) =$

$\mathbb{1}_{\{i\in\mathcal{N}_r\}} - \mathbb{1}_{\{j\in\mathcal{N}_r\}}$. (f) Taking χ, β, and $\widetilde{\Omega}_\ell$ as given: $\frac{\partial \log TFP_r}{\partial \widetilde{\Omega}^x_{im}} - \frac{\partial \log TFP_r}{\partial \widetilde{\Omega}^x_{ij}} = \frac{\partial \log \Phi_r}{\partial \widetilde{\Omega}^x_{im}} - \frac{\partial \log \Phi_r}{\partial \widetilde{\Omega}^x_{ij}} - \mu_i \lambda_i (F^r_m - F^r_j) = \frac{\mu_i\lambda_i}{\Phi_r}(\psi_{m|r} - \psi_{j|r} + \mathbb{1}_{\{i\notin\mathcal{N}_r\}}(\mathbb{1}_{\{m\in\mathcal{N}_r, j\notin\mathcal{N}_r\}}) - \mathbb{1}_{\{j\in\mathcal{N}_r, m\notin\mathcal{N}_r\}}) - \mathbb{1}_{\{i\in\mathcal{N}_r\}}(\mathbb{1}_{\{j\in\mathcal{N}_r, m\notin\mathcal{N}_r\}}) - \mathbb{1}_{\{m\in\mathcal{N}_r, j\notin\mathcal{N}_r\}})) + \mu_i\lambda_i(F^r_j - F^r_m)$, where $\mathbb{1}_{\{i\notin\mathcal{N}_r\}}(\mathbb{1}_{\{m\in\mathcal{N}_r, j\notin\mathcal{N}_r\}}) - \mathbb{1}_{\{j\in\mathcal{N}_r, m\notin\mathcal{N}_r\}}) - \mathbb{1}_{\{i\in\mathcal{N}_r\}}(\mathbb{1}_{\{j\in\mathcal{N}_r, m\notin\mathcal{N}_r\}}) - \mathbb{1}_{\{m\in\mathcal{N}_r, j\notin\mathcal{N}_r\}}) = \mathbb{1}_{\{m\in\mathcal{N}_r\}} - \mathbb{1}_{\{j\in\mathcal{N}_r\}}$.

1.6.2 Proof for Corollary 2

Introduce in (22) $d\,\Phi_r = \sum_{h\in\mathcal{H}_r} d\,\Lambda_h + \sum_{i\in\mathcal{N}_r}((1-\mu_i)d\,\lambda_i - \lambda_i\,d\,\mu_i)$ and the first-order variations for Λ and λ from Theorem 1 in Rojas-Bernal (2026) to obtain

$$d\log Y_r = \sum_{i\in\mathcal{N}} \ddot{\lambda}^r_i\, d\log A_i + \sum_{i\in\mathcal{N}_r}(1-\mu_i)\frac{\lambda_i}{\Phi_r}\Big(d\log(\mu_i\lambda_i) + \sum_{m\in\mathcal{N}} \widetilde{\Omega}^x_{im} \sum_{j\in\mathcal{N}} \widetilde{\psi}^x_{mj}\, d\log\mu_j\Big)$$
$$-\sum_{i\in\mathcal{N}_r}(1-\mu_i)\frac{\lambda_i}{\Phi_r}\Big(\sum_{h\in\mathcal{H}_r}\widetilde{\Omega}^\ell_{ih}\, d\log\Lambda_h + \sum_{h\in\mathcal{H}}\sum_{j\in\mathcal{N}}\widetilde{\Omega}^x_{ij}\,\widetilde{\psi}^\ell_{jh}\, d\log\Lambda_h\Big) + \sum_{h\in\mathcal{H}}\ddot{\Lambda}^r_h\, d\log L_h.$$

1.7 Example 1

$\widetilde{\Omega}_\ell = \begin{pmatrix} 1-\omega_e \\ 1-\omega_f \end{pmatrix}$, $\widetilde{\Omega}_x = \begin{pmatrix} 0 & \omega_e \\ \omega_f & 0 \end{pmatrix}$, $\Omega_\ell = \begin{pmatrix} \mu_e(1-\omega_e) \\ \mu_f(1-\omega_f) \end{pmatrix}$, and $\Omega_x = \begin{pmatrix} 0 & \mu_e\omega_e \\ \mu_f\omega_f & 0 \end{pmatrix}$.

Then

$$\widetilde{\Psi}_x = \begin{pmatrix} \widetilde{\psi}^x_{ee} & \widetilde{\psi}^x_{ef} \\ \widetilde{\psi}^x_{fe} & \widetilde{\psi}^x_{ff} \end{pmatrix} = \begin{pmatrix} \frac{1}{1-\omega_e\omega_f} & \frac{\omega_e}{1-\omega_e\omega_f} \\ \frac{\omega_f}{1-\omega_e\omega_f} & \frac{1}{1-\omega_e\omega_f} \end{pmatrix},$$

$$\Psi_x = \begin{pmatrix} \psi^x_{ee} & \psi^x_{ef} \\ \psi^x_{fe} & \psi^x_{ff} \end{pmatrix} = \begin{pmatrix} \frac{1}{1-\mu_e\omega_e\mu_f\omega_f} & \frac{\mu_e\omega_e}{1-\mu_e\omega_e\mu_f\omega_f} \\ \frac{\mu_f\omega_f}{1-\mu_e\omega_e\mu_f\omega_f} & \frac{1}{1-\mu_e\omega_e\mu_f\omega_f} \end{pmatrix},$$

$$\widetilde{\Psi}_\ell = \begin{pmatrix} \widetilde{\psi}^\ell_e \\ \widetilde{\psi}^\ell_f \end{pmatrix} = \widetilde{\Psi}_x\widetilde{\Omega}_\ell = \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \quad \Psi_\ell = \begin{pmatrix} \psi^\ell_e \\ \psi^\ell_f \end{pmatrix} = \Psi_x\Omega_\ell = \begin{pmatrix} \frac{\mu_e(1-\omega_e(1-\mu_f(1-\omega_f)))}{1-\mu_e\omega_e\mu_f\omega_f} \\ \frac{\mu_f(1-\omega_f(1-\mu_e(1-\omega_e)))}{1-\mu_e\omega_e\mu_f\omega_f} \end{pmatrix},$$

$$\lambda = \begin{pmatrix} \lambda_e \\ \lambda_f \end{pmatrix} = \Psi'_x \begin{pmatrix} \beta \\ 1-\beta \end{pmatrix} = \begin{pmatrix} \frac{\beta+(1-\beta)\mu_f\omega_f}{1-\mu_e\omega_e\mu_f\omega_f} \\ \frac{1-\beta+\beta\mu_e\omega_e}{1-\mu_e\omega_e\mu_f\omega_f} \end{pmatrix},$$

$$\Lambda = \frac{\beta\mu_e(1-\omega_e(1-\mu_f(1-\omega_f))) + (1-\beta)\mu_f(1-\omega_f(1-\mu_e(1-\omega_e)))}{1-\mu_e\omega_e\mu_f\omega_f},$$

$$1 = (1-\mu_e\omega_e)\lambda_e + (1-\mu_f\omega_f)\lambda_f,$$

$$\Phi_e = (1-\mu_e\omega_e)\lambda_e, \qquad \Phi_f = (1-\mu_f\omega_f)\lambda_f, \qquad \ddot{\Lambda}^e = \ddot{\Lambda}^f = 1,$$

$$\ddot{\lambda}^e_e = \frac{\lambda_e}{\Phi_e} + \frac{\lambda_e}{\Phi_e}\widetilde{\psi}^x_{ee,\Delta\Omega} = \frac{\lambda_e}{\Phi_e}(1+(1-\mu_e)\omega_e\widetilde{\psi}^x_{fe}) = \frac{1-\mu_e\omega_e\omega_f}{(1-\mu_e\omega_e)(1-\omega_e\omega_f)},$$

$$\ddot{\lambda}^e_f = \frac{\lambda_e}{\Phi_e}\widetilde{\psi}^x_{ef,\Delta\Omega} = \frac{\lambda_e}{\Phi_e}(1-\mu_e)\omega_e\widetilde{\psi}^x_{ff} = \frac{(1-\mu_e)\omega_e}{(1-\mu_e\omega_e)(1-\omega_e\omega_f)},$$

$$\ddot{\lambda}_f^f = \frac{\lambda_f}{\Phi_f} + \frac{\lambda_f}{\Phi_f}\widetilde{\psi}^x_{ff,\Delta\Omega} = \frac{\lambda_f}{\Phi_f}(1 + (1-\mu_f)\omega_f\widetilde{\psi}^x_{ef}) = \frac{1-\mu_f\omega_f\omega_e}{(1-\mu_f\omega_f)(1-\omega_e\omega_f)},$$

$$\ddot{\lambda}_e^f = \frac{\lambda_f}{\Phi_f}\widetilde{\psi}^x_{fe,\Delta\Omega} = \frac{\lambda_f}{\Phi_f}(1-\mu_f)\omega_f\widetilde{\psi}^x_{ee} = \frac{(1-\mu_f)\omega_f}{(1-\mu_f\omega_f)(1-\omega_e\omega_f)},$$

$$\begin{aligned}\Lambda &= (1-\omega_e)\mu_e\lambda_e + (1-\omega_f)\mu_f\lambda_f \\ &= \frac{\beta\mu_e(1-\omega_e+\mu_f\omega_e(1-\omega_f)) + (1-\beta)\mu_f(1-\omega_f+\mu_e\omega_f(1-\omega_e))}{1-\mu_e\omega_e\mu_f\omega_f},\end{aligned}$$

$$F_e = F_e^e = F_e^f = \frac{\psi_e^\ell}{\Lambda} = \frac{\mu_e(1-\omega_e(1-\mu_f(1-\omega_f)))}{\Lambda(1-\mu_e\omega_e\mu_f\omega_f)},$$

$$F_f = F_f^e = F_f^f = \frac{\psi_f^\ell}{\Lambda} = \frac{\mu_f(1-\omega_f(1-\mu_e(1-\omega_e)))}{\Lambda(1-\mu_e\omega_e\mu_f\omega_f)},$$

$$\psi_{e|e} = \psi^x_{ef}\Omega^x_{fe} - \psi^x_{ee}\Omega^x_{ef} = -\frac{\mu_e\omega_e(1-\mu_f\omega_f)}{1-\mu_e\omega_e\mu_f\omega_f},$$

$$\psi_{f|e} = \psi^x_{ff}\Omega^x_{fe} - \psi^x_{fe}\Omega^x_{ef} = \frac{\mu_f\omega_f(1-\mu_e\omega_e)}{1-\mu_e\omega_e\mu_f\omega_f},$$

$$\psi_{e|f} = \psi^x_{ee}\Omega^x_{ef} - \psi^x_{ef}\Omega^x_{fe} = \frac{\mu_e\omega_e(1-\mu_f\omega_f)}{1-\mu_e\omega_e\mu_f\omega_f},$$

$$\psi_{f|f} = \psi^x_{fe}\Omega^x_{ef} - \psi^x_{ff}\Omega^x_{fe} = -\frac{\mu_f\omega_f(1-\mu_e\omega_e)}{1-\mu_e\omega_e\mu_f\omega_f},$$

1.8 Example 2

$\widetilde{\Omega}_\ell = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ 0 & \frac{1}{2} \end{pmatrix}$, $\widetilde{\Omega}_x = \begin{pmatrix} 0 & 0 \\ \frac{1}{2} & 0 \end{pmatrix}$, $\Omega_\ell = \begin{pmatrix} \frac{\mu}{2} & \frac{\mu}{2} \\ 0 & \frac{\mu}{2} \end{pmatrix}$, $\Omega_x = \begin{pmatrix} 0 & 0 \\ \frac{\mu}{2} & 0 \end{pmatrix}$,$\beta = \begin{pmatrix} \beta_h & 1-\beta_h \\ \beta_l & 1-\beta_l \end{pmatrix}$,

$$\widetilde{\Psi}_x = \begin{pmatrix} \widetilde{\psi}^x_{mm} & \widetilde{\psi}^x_{ma} \\ \widetilde{\psi}^x_{am} & \widetilde{\psi}^x_{aa} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ \frac{1}{2} & 1 \end{pmatrix}, \qquad \Psi_x = \begin{pmatrix} \psi^x_{mm} & \psi^x_{ma} \\ \psi^x_{am} & \psi^x_{aa} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ \frac{\mu}{2} & 1 \end{pmatrix},$$

$$\widetilde{\Psi}_\ell = \begin{pmatrix} \widetilde{\psi}^\ell_{mh} & \widetilde{\psi}^\ell_{ml} \\ \widetilde{\psi}^\ell_{ah} & \widetilde{\psi}^\ell_{al} \end{pmatrix} = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{4} & \frac{3}{4} \end{pmatrix}, \quad \Psi_\ell = \begin{pmatrix} \psi^\ell_{mh} & \psi^\ell_{ml} \\ \psi^\ell_{ah} & \psi^\ell_{al} \end{pmatrix} = \begin{pmatrix} \frac{\mu}{2} & \frac{\mu}{2} \\ \frac{\mu^2}{4} & \left(1+\frac{\mu}{2}\right)\frac{\mu}{2} \end{pmatrix},$$

$$\mathscr{B} = \begin{pmatrix} \mathscr{B}_{hm} & \mathscr{B}_{ha} \\ \mathscr{B}_{lm} & \mathscr{B}_{la} \end{pmatrix} = \begin{pmatrix} \beta_h + (1-\beta_h)\frac{\mu}{2} & (1-\beta_h) \\ \beta_l + (1-\beta_l)\frac{\mu}{2} & (1-\beta_l) \end{pmatrix},$$

$$\mathscr{C} = \begin{pmatrix} \mathscr{C}_{hh} & \mathscr{C}_{hl} \\ \mathscr{C}_{lh} & \mathscr{C}_{ll} \end{pmatrix} = \begin{pmatrix} (\beta_h + (1-\beta_h)\frac{\mu}{2})\frac{\mu}{2} & \left(\beta_h + (1-\beta_h)\left(1+\frac{\mu}{2}\right)\right)\frac{\mu}{2} \\ (\beta_l + (1-\beta_l)\frac{\mu}{2})\frac{\mu}{2} & \left(\beta_l + (1-\beta_l)\left(1+\frac{\mu}{2}\right)\right)\frac{\mu}{2} \end{pmatrix}.$$

In equilibrium $\chi_l = 1-\chi_h$

$$(i)\quad \lambda_m = \beta_l + (1-\beta_l)\frac{\mu}{2} + (\beta_h-\beta_l)\left(1-\frac{\mu}{2}\right)\chi_h,$$
$$(ii)\quad \lambda_a = 1-\beta_l-(\beta_h-\beta_l)\chi_h,$$
$$(iii)\quad \Lambda_h = \left(\frac{\mu}{2}+\beta_l\left(1-\frac{\mu}{2}\right)\right)\frac{\mu}{2} + (\beta_h-\beta_l)\left(1-\frac{\mu}{2}\right)\frac{\mu}{2}\chi_h,$$
$$(iv)\quad \Lambda_l = \left(1+(1-\beta_l)\frac{\mu}{2}\right)\frac{\mu}{2} - (\beta_h-\beta_l)\frac{\mu^2}{4}\chi_h,$$
$$(v)\quad \chi_h = \Lambda_h + \frac{1}{2}(1-\mu)\left(\lambda_m+\lambda_a\right).$$

Solving the system gives

$$\chi_h = \frac{2-(1-\beta_l)\mu}{4-(\beta_h-\beta_l)\mu},\quad \lambda_m = \frac{\beta_h+\beta_l+(1-\beta_h)\mu}{2-(\beta_h-\beta_l)\frac{\mu}{2}},\quad \lambda_a = \frac{2-\beta_h-\beta_l}{2-(\beta_h-\beta_l)\frac{\mu}{2}},$$

$$\Lambda_h = \frac{\mu}{2}\left(\frac{\beta_h+\beta_l+(1-\beta_h)\mu}{2-(\beta_h-\beta_l)\frac{\mu}{2}}\right),\quad \Lambda_l = \frac{\mu}{2}\left(\frac{2+(1-\beta_h)\mu}{2-(\beta_h-\beta_l)\frac{\mu}{2}}\right),$$

$$\Phi_m = \lambda_m,\quad \Phi_a = \left(1-\frac{\mu}{2}\right)\lambda_a,\quad \ddot{\lambda}^m_m = \frac{\lambda_m}{\Phi_m} = 1,\quad \ddot{\lambda}^m_a = 0,$$

$$\ddot{\lambda}^a_a = \frac{\lambda_a}{\Phi_a} = \frac{2}{2-\mu},\quad \ddot{\lambda}^a_m = \frac{1-\mu}{2-\mu},\quad \ddot{\Lambda}^m_h = \frac{1}{2}\frac{\lambda_m}{\Phi_m} = \frac{1}{2},\quad \ddot{\Lambda}^m_l = \frac{1}{2}\frac{\lambda_m}{\Phi_m} = \frac{1}{2},$$

$$\ddot{\Lambda}^a_h = \frac{1-\mu}{4}\frac{\lambda_a}{\Phi_a} = \frac{1-\mu}{4-2\mu},\quad \ddot{\Lambda}^a_l = \frac{1}{2}\frac{\lambda_a}{\Phi_a}\left(1+(1-\mu)\frac{1}{2}\right) = \frac{3-\mu}{4-2\mu},$$

$$\delta^m_h = \frac{\ddot{\Lambda}^m_h}{\Lambda_h} = \frac{2-(\beta_h-\beta_l)\frac{\mu}{2}}{\mu(\beta_h+\beta_l+(1-\beta_h)\mu)},\quad \delta^m_l = \frac{\ddot{\Lambda}^m_l}{\Lambda_l} = \frac{2-(\beta_h-\beta_l)\frac{\mu}{2}}{\mu(2+(1-\beta_h)\mu)},$$

$$\delta^a_h = \frac{\ddot{\Lambda}^a_h}{\Lambda_h} = \frac{(1-\mu)\left(2-(\beta_h-\beta_l)\frac{\mu}{2}\right)}{\mu(2-\mu)(\beta_h+\beta_l+(1-\beta_h)\mu)},\quad \delta^a_l = \frac{\ddot{\Lambda}^a_l}{\Lambda_l} = \frac{(3-\mu)\left(2-(\beta_h-\beta_l)\frac{\mu}{2}\right)}{\mu(2-\mu)(2+(1-\beta_h)\mu)},$$

$$F^m_m = \left(1-(\beta_h-\beta_l)\frac{\mu}{4}\right)\left(\frac{1}{(\beta_h+\beta_l+(1-\beta_h)\mu)}+\frac{1}{(2+(1-\beta_h)\mu)}\right),$$

$$F^m_a = \frac{1}{2}\left(1-(\beta_h-\beta_l)\frac{\mu}{4}\right)\left(\frac{\mu}{\beta_h+\beta_l+(1-\beta_h)\mu}+\frac{(2+\mu)}{2+(1-\beta_h)\mu}\right),$$

$$F^a_m = \frac{\left(1-(\beta_h-\beta_l)\frac{\mu}{4}\right)}{(2-\mu)}\left(\frac{(1-\mu)}{\beta_h+\beta_l+(1-\beta_h)\mu}+\frac{(3-\mu)}{2+(1-\beta_h)\mu}\right),$$

$$F^a_a = \frac{1}{2}\frac{\left(1-(\beta_h-\beta_l)\frac{\mu}{4}\right)}{(2-\mu)}\left(\frac{\mu(1-\mu)}{\beta_h+\beta_l+(1-\beta_h)\mu}+\frac{(2+\mu)(3-\mu)}{2+(1-\beta_h)\mu}\right),$$

$$F^m_m - F^m_a = \frac{1}{2}\left(1-(\beta_h-\beta_l)\frac{\mu}{4}\right)\left(\frac{2-\mu}{\beta_h+\beta_l+(1-\beta_h)\mu}-\frac{\mu}{2+(1-\beta_h)\mu}\right),$$

$$F^a_m - F^a_a = \frac{1}{2}\frac{\left(1-(\beta_h-\beta_l)\frac{\mu}{4}\right)}{(2-\mu)}\left(\frac{(1-\mu)(2-\mu)}{\beta_h+\beta_l+(1-\beta_h)\mu}-\frac{(3-\mu)\mu}{2+(1-\beta_h)\mu}\right),$$

$$M^m_h = \left(1-(\beta_h-\beta_l)\frac{\mu}{4}\right)\left(\frac{\left(\beta_h+(1-\beta_h)\frac{\mu}{2}\right)}{\beta_h+\beta_l+(1-\beta_h)\mu}+\frac{\left(\beta_h+(1-\beta_h)\left(1+\frac{\mu}{2}\right)\right)}{2+(1-\beta_h)\mu}\right),$$

$$M_l^m = \left(1 - (\beta_h - \beta_l)\frac{\mu}{4}\right)\left(\frac{(\beta_l + (1-\beta_l)\frac{\mu}{2})}{\beta_h + \beta_l + (1-\beta_h)\mu} + \frac{(\beta_l + (1-\beta_l)\left(1+\frac{\mu}{2}\right))}{2 + (1-\beta_h)\mu}\right),$$

$$M_h^a = \left(1 - (\beta_h - \beta_l)\frac{\mu}{4}\right)\left(\frac{(1-\mu)\left(\beta_h + (1-\beta_h)\frac{\mu}{2}\right)}{(2-\mu)(\beta_h + \beta_l + (1-\beta_h)\mu)} + \frac{(3-\mu)\left(\beta_h + (1-\beta_h)\left(1+\frac{\mu}{2}\right)\right)}{(2-\mu)(2 + (1-\beta_h)\mu)}\right),$$

$$M_l^a = \left(1 - (\beta_h - \beta_l)\frac{\mu}{4}\right)\left(\frac{(1-\mu)\left(\beta_l + (1-\beta_l)\frac{\mu}{2}\right)}{(2-\mu)(\beta_h + \beta_l + (1-\beta_h)\mu)} + \frac{(3-\mu)\left(\beta_l + (1-\beta_l)\left(1+\frac{\mu}{2}\right)\right)}{(2-\mu)(2 + (1-\beta_h)\mu)}\right),$$

$$M_h^m - M_l^m = \frac{1}{8}(\beta_h - \beta_l)\left(4 - (\beta_h - \beta_l)\mu\right)\left(\frac{2-\mu}{\beta_h + \beta_l + (1-\beta_h)\mu} - \frac{\mu}{2 + (1-\beta_h)\mu}\right),$$

$$M_h^a - M_l^a = \frac{1}{8}(\beta_h - \beta_l)(4 - (\beta_h - \beta_l)\mu)\left(\frac{1-\mu}{\beta_h + \beta_l + (1-\beta_h)\mu} - \frac{(3-\mu)\mu}{(2-\mu)(2 + (1-\beta_h)\mu)}\right),$$

$$\mathscr{B}_{h|m} = \mathscr{B}_{ha}\Omega^x_{am} - \mathscr{B}_{hm}\Omega^x_{ma} = (1-\beta_h)\frac{\mu}{2}, \quad \mathscr{B}_{l|m} = \mathscr{B}_{la}\Omega^x_{am} - \mathscr{B}_{lm}\Omega^x_{ma} = (1-\beta_l)\frac{\mu}{2},$$

$$\mathscr{B}_{h|a} = \mathscr{B}_{hm}\Omega^x_{ma} - \mathscr{B}_{ha}\Omega^x_{am} = -(1-\beta_h)\frac{\mu}{2}, \quad \mathscr{B}_{l|a} = \mathscr{B}_{lm}\Omega^x_{ma} - \mathscr{B}_{la}\Omega^x_{am} = -(1-\beta_l)\frac{\mu}{2},$$

$$\psi_{m|m} = \psi^x_{ma}\Omega^x_{am} - \psi^x_{mm}\Omega^x_{ma} = 0, \quad \psi_{a|m} = \psi^x_{aa}\Omega^x_{am} - \psi^x_{am}\Omega^x_{ma} = \frac{\mu}{2},$$

$$\psi_{m|a} = \psi^x_{mm}\Omega^x_{ma} - \psi^x_{ma}\Omega^x_{am} = 0, \quad \psi_{a|a} = \psi^x_{am}\Omega^x_{ma} - \psi^x_{aa}\Omega^x_{am} = -\frac{\mu}{2}.$$

Then

$$\begin{aligned} d\log Y_m &= d\log A_m + (1 - \lambda_m F_m^m)\, d\log\mu_m + \lambda_a\left(\frac{\mu}{\Phi_m} - F_a^m\right) d\log\mu_a + \frac{1}{2}\left(d\log L_h + d\log L_l\right) \\ &+ \left(\left(1 - \frac{\mu}{2}\right)\frac{\Delta\beta}{\Phi_m} - (M_h^m - M_l^m)\right) d\chi_h + \left(\frac{1}{\Phi_m}\left(1 - \frac{\mu}{2}\right) - (F_m^m - F_a^m)\right)(\chi_h d\beta_h + \chi_l d\beta_l), \end{aligned}$$

$$\begin{aligned} d\log Y_a &= \frac{2}{2-\mu} d\log A_a + \frac{1-\mu}{2-\mu} d\log A_m + \left(\frac{2}{2-\mu} - \lambda_a\left(\frac{\mu}{\Phi_a} + F_a^a\right)\right) d\log\mu_a \\ &+ \left(\frac{1-\mu}{2-\mu} - \lambda_m F_m^a\right) d\log\mu_m + \left((M_l^a - M_h^a) - \left(1 - \frac{\mu}{2}\right)\frac{\Delta\beta}{\Phi_a}\right) d\chi_h \\ &+ \left((F_a^a - F_m^a) - \frac{1}{\Phi_a}\left(1 - \frac{\mu}{2}\right)\right)(\chi_h d\beta_h + \chi_l d\beta_l) + \frac{1-\mu}{4-2\mu} d\log L_h + \frac{3-\mu}{4-2\mu} d\log L_l. \end{aligned}$$

2 Relationship with Baqaee & Farhi (2020, 2024)

2.1 The Environment

Baqaee & Farhi (2024) develop an open-economy production-network model with heterogeneous households and firm-level distortions, modeled as wedges between prices and marginal costs. Baqaee & Farhi (2020) is the corresponding closed-economy rspecial case with a representative household. The environment is characterized by: (i) a set of countries $\mathscr{R}$, a set of producers $\mathscr{N}$, and a set of factors $\mathscr{F}$; (ii) the producers located in r are $\mathscr{N}_r \subseteq \mathscr{N}$; (iii) the factors located in in r are $\mathscr{F}_r \subseteq \mathscr{F}$; (iv) country r has a representative household that supplies factors and "*fictitious factors*" , which represent rebated markup (wedge) revenue; ownership is summarized by an $R \times F$ matrix Φ, where Φ_{rf} where Φ_{rf} is the share of factor f's income accruing to household r; and (v) primary factor supplies are exogenous.

2.2 Notational equivalences

From equation (20)

$$\begin{aligned}
\widehat{p}_{Y_r} &= \sum_{i\in\mathscr{N}_r} \frac{\lambda_i}{\Phi_r}\Big(\widehat{p}_i - \sum_{j\in\mathscr{N}} \Omega^x_{ij}\widehat{p}_j\Big) = \sum_{i\in\mathscr{N}_r} \frac{\lambda_i}{\Phi_r}\widehat{p}_i - \sum_{i\in\mathscr{N}_r} \frac{\lambda_i}{\Phi_r}\sum_{j\in\mathscr{N}_r}\Omega^x_{ij}\widehat{p}_j - \sum_{i\in\mathscr{N}_r} \frac{\lambda_i}{\Phi_r}\sum_{j\notin\mathscr{N}_r}\Omega^x_{ij}\widehat{p}_j \\
&= \sum_{i\in\mathscr{N}_r}\Big(\frac{\lambda_i}{\Phi_r} - \sum_{j\in\mathscr{N}_r}\Omega^x_{ji}\frac{\lambda_j}{\Phi_r}\Big)\widehat{p}_i - \sum_{i\notin\mathscr{N}_r}\Big(\sum_{j\in\mathscr{N}_r}\Omega^x_{ji}\frac{\lambda_j}{\Phi_r}\Big)\widehat{p}_i \\
&= \sum_{i\in\mathscr{N}_r}\Big(\frac{S_i}{GDP_r} - \sum_{j\in\mathscr{N}_r}\frac{p_i x_{ji}}{S_j}\frac{S_j}{GDP_r}\Big)\widehat{p}_i - \sum_{i\notin\mathscr{N}_r}\Big(\sum_{j\in\mathscr{N}_r}\frac{p_i x_{ji}}{S_j}\frac{S_j}{GDP_r}\Big)\widehat{p}_i \\
&= \sum_{i\in\mathscr{N}_r}\frac{p_i}{GDP_r}\Big(y_i - \sum_{j\in\mathscr{N}_r}x_{ji}\Big)\widehat{p}_i - \sum_{i\notin\mathscr{N}_r}\frac{p_i}{GDP_r}\Big(\sum_{j\in\mathscr{N}_r}x_{ji}\Big)\widehat{p}_i = \sum_{i\in\mathscr{N}}\Omega_{Y_r i}\,\widehat{p}_i.
\end{aligned}$$

This is the country-level Divisia GDP deflator in Baqaee & Farhi (2024).

Now, from equation (21)

$$\begin{aligned}
\widehat{Y}_r &= \sum_{i\in\mathscr{N}_r}\frac{\lambda_i}{\Phi_r}\Big(\widehat{y}_i - \sum_{j\in\mathscr{N}}\Omega^x_{ij}\widehat{x}_{ij}\Big) = \sum_{i\in\mathscr{N}_r}\Big(\frac{\lambda_i}{\Phi_r}\widehat{y}_i - \sum_{j\in\mathscr{N}_r}\Omega^x_{ji}\frac{\lambda_j}{\Phi_r}\widehat{x}_{ji}\Big) - \sum_{i\notin\mathscr{N}_r}\sum_{j\in\mathscr{N}_r}\Omega^x_{ji}\frac{\lambda_j}{\Phi_r}\widehat{x}_{ji} \\
&= \sum_{i\in\mathscr{N}_r}\Omega_{Y_r i}\Big(\frac{S_i}{p_i q_{ri}}\widehat{y}_i - \sum_{j\in\mathscr{N}_r}\frac{p_i x_{ji}}{p_i q_{ri}}\widehat{x}_{ji}\Big) - \sum_{i\notin\mathscr{N}_r}\Omega_{Y_r i}\sum_{j\in\mathscr{N}_r}\frac{p_i x_{ji}}{p_i q_{ri}}\widehat{x}_{ji} \\
&= \sum_{i\in\mathscr{N}_r}\Omega_{Y_r i}\widehat{q}_{ri} - \sum_{i\notin\mathscr{N}_r}\Omega_{Y_r i}\widehat{q}_{ri} = \sum_{i\in\mathscr{N}}\Omega_{Y_r i}\widehat{q}_{ri}.
\end{aligned}$$

This coincides with the country-level Divisia real GDP in Baqaee & Farhi (2024).

2.3 First order approximation for country-level GDP

From equation (18), the first-order approximation to GDP is

$$\widehat{GDP}_r = \widehat{p}_{Y_r} + \widehat{Y}_r = \sum_{i\in\mathscr{N}_r}\frac{\lambda_i}{\Phi_r}\sum_{h\in\mathscr{H}}\Omega^{\ell}_{ih}(\widehat{w}_h + \widehat{\ell}_{ih}) + \sum_{i\in\mathscr{N}_r}\frac{\lambda_i}{\Phi_r}\big((1-\mu_i)\widehat{S}_i - \mu_i\widehat{\mu}_i\big).$$

Now, introducing $\widehat{p} = \widetilde{\Psi}_x(\widetilde{\Omega}_\ell\, \widehat{w} - \widehat{\mathcal{A}} - \widehat{\mu})$ in equation (20)

$$
\begin{aligned}
\widehat{p}_{Y_r} &= \sum_{i\in\mathcal{N}} \Omega_{Y_r i}\, \widehat{p}_i = \sum_{i\in\mathcal{N}_r} \Omega_{Y_r i}\, \widehat{p}_i + \sum_{i\notin\mathcal{N}_r} \Omega_{Y_r i}\, \widehat{p}_i \\
&= \sum_{i\in\mathcal{N}_r} \Omega_{Y_r i}\Big(\sum_{j\in\mathcal{N}_r} \widetilde{\psi}^{x_r}_{ij} \sum_{h\in\mathcal{H}} \widetilde{\Omega}^{\ell}_{jh}\widehat{w}_h + \sum_{j\in\mathcal{N}_r} \widetilde{\psi}^{x_r}_{ij} \sum_{i\notin\mathcal{N}_r} \widetilde{\Omega}^{x}_{ji}\widehat{p}_i - \sum_{j\in\mathcal{N}_r} \widetilde{\psi}^{x}_{ij}(\widehat{\mathcal{A}}_j + \widehat{\mu}_j)\Big) + \sum_{i\notin\mathcal{N}_r} \Omega_{Y_r i}\, \widehat{p}_i \\
&= \sum_{h\in\mathcal{H}} \widetilde{\Lambda}_{Y_r h}\widehat{w}_h - \sum_{j\in\mathcal{N}_r} \widetilde{\lambda}_{Y_r j}(\widehat{\mathcal{A}}_j + \widehat{\mu}_j) + \sum_{i\notin\mathcal{N}_r} \widetilde{\Lambda}_{Y_r i}\widehat{p}_i + \sum_{i\notin\mathcal{N}_r} \Omega_{Y_r i}\, \widehat{p}_i.
\end{aligned}
$$

Then

$$
\begin{aligned}
\widehat{Y}_r &= \sum_{i\in\mathcal{N}_r} \frac{\lambda_i}{\Phi_r} \sum_{h\in\mathcal{H}} \Omega^{\ell}_{ih}(\widehat{w}_h + \widehat{\ell}_{ih}) + \sum_{i\in\mathcal{N}_r} \frac{\lambda_i}{\Phi_r}((1-\mu_i)\widehat{S}_i - \mu_i\widehat{\mu}_i) - \widehat{p}_{Y_r} \\
&= \sum_{i\in\mathcal{N}_r} \frac{\lambda_i}{\Phi_r} \sum_{h\in\mathcal{H}} \Omega^{\ell}_{ih}(\widehat{w}_h + \widehat{\ell}_{ih}) + \sum_{i\in\mathcal{N}_r} \frac{\lambda_i}{\Phi_r}((1-\mu_i)\widehat{S}_i - \mu_i\widehat{\mu}_i) \\
&\quad - \sum_{h\in\mathcal{H}} \widetilde{\Lambda}_{Y_r h}\widehat{w}_h + \sum_{j\in\mathcal{N}_r} \widetilde{\lambda}_{Y_r j}(\widehat{\mathcal{A}}_j + \widehat{\mu}_j) - \sum_{i\notin\mathcal{N}_r} \widetilde{\Lambda}_{Y_r i}\widehat{p}_i - \sum_{i\notin\mathcal{N}_r} \Omega_{Y_r i}\, \widehat{p}_i \\
&= \sum_{i\in\mathcal{N}_r} \frac{\lambda_i}{\Phi_r} \sum_{h\in\mathcal{H}} \Omega^{\ell}_{ih}(\widehat{w}_h + \widehat{\ell}_{fh}) + \sum_{i\in\mathcal{N}_r} \frac{\lambda_i}{\Phi_r}((1-\mu_i)\widehat{S}_i - \mu_i\widehat{\mu}_i) \\
&\quad - \sum_{h\in\mathcal{H}} \widetilde{\Lambda}_{Y_r h}\widehat{w}_h + \sum_{j\in\mathcal{N}_r} \widetilde{\lambda}_{Y_r j}(\widehat{\mathcal{A}}_j + \widehat{\mu}_j) - \sum_{i\notin\mathcal{N}_r} (\Omega_{Y_r i} + \widetilde{\Lambda}_{Y_r i})(\widehat{\Omega}_{Y_r i} - \widehat{q}_{ri} + \widehat{GDP}_r) \\
&= \sum_{i\in\mathcal{N}_r}\sum_{h\in\mathcal{H}} \frac{J_{ih}}{GDP_r}\widehat{J}_{ih} + \sum_{i\in\mathcal{N}_r} \lambda_{Y_r i}((1-\mu_i)\widehat{S}_i - \mu_i\widehat{\mu}_i) - \sum_{h\in\mathcal{H}} \widetilde{\Lambda}_{Y_r h}\widehat{J}_h + \sum_{h\in\mathcal{H}} \widetilde{\Lambda}_{Y_r h}\widehat{L}_h \\
&\quad + \sum_{j\in\mathcal{N}_r} \widetilde{\lambda}_{Y_r j}(\widehat{\mathcal{A}}_j + \widehat{\mu}_j) - \sum_{i\notin\mathcal{N}_r} (\Omega_{Y_r i} + \widetilde{\Lambda}_{Y_r i})(\widehat{\Omega}_{Y_r i} - \widehat{q}_{ri} + \widehat{GDP}_r) \\
&= \sum_{h\in\mathcal{H}} \Lambda_{Y_r h}\widehat{\Lambda}_{Y_r h} + \sum_{i\in\mathcal{N}_r} \lambda_{Y_r i}((1-\mu_i)\widehat{\lambda}_{Y_r i} - \mu_i\widehat{\mu}_i) - \sum_{h\in\mathcal{H}} \widetilde{\Lambda}_{Y_r h}\widehat{J}_h + \sum_{h\in\mathcal{H}} \widetilde{\Lambda}_{Y_r h}\widehat{L}_h \\
&\quad + \sum_{j\in\mathcal{N}_r} \widetilde{\lambda}_{Y_r j}(\widehat{\mathcal{A}}_j + \widehat{\mu}_j) + \sum_{i\notin\mathcal{N}_r} (\Omega_{Y_r i} + \widetilde{\Lambda}_{Y_r i})(\widehat{q}_{ri} - \widehat{\Omega}_{Y_r i}) \\
&\quad + \Big(\sum_{h\in\mathcal{H}} \Lambda_{Y_r h} + \sum_{i\in\mathcal{N}_r} (1-\mu_i)\lambda_{Y_r i} - \sum_{i\notin\mathcal{N}_r} (\Omega_{Y_r i} + \sum_{j\in\mathcal{N}_r} \widetilde{\lambda}_{Y_r j}\widetilde{\Omega}^{x}_{jf})\Big)\widehat{GDP}_r.
\end{aligned}
$$

Let me assume, as in their paper, that factor markets are segmented by country. This comes from their definition $GDP_r = \sum_{i\in\mathcal{N}_r}(\sum_{h\in\mathcal{H}_r} w_h\, \ell_{ih} + (1-\mu_i)\, S_i)$. Hence $\Lambda_{Y_r h} = \mathbb{1}\,\{h \in \mathcal{H}_r\}\, \frac{J_h}{GDP_r}$ and $\widetilde{\Lambda}_{Y_r h} = \mathbb{1}\,\{h \in \mathcal{H}_r\} \sum_{i\in\mathcal{N}_r} \Omega_{Y_r i} \sum_{j\in\mathcal{N}_r} \widetilde{\psi}^{x_r}_{ij}\, \widetilde{\Omega}^{\ell}_{jh}$. Then, it follows that $\sum_{h\in\mathcal{H}_r} \Lambda_{Y_r h} + \sum_{i\in\mathcal{N}_r}(1-\mu_i)\lambda_{Y_r i} = 1$ and $\sum_{h\in\mathcal{H}_r} \Lambda_{Y_r h}\widehat{\Lambda}_{Y_r h} + \sum_{i\in\mathcal{N}_r} \lambda_{Y_r i}((1-\mu_i)\widehat{\lambda}_{Y_r i} - \mu_i\widehat{\mu}_i) = 0$. Therefore

$$
\begin{aligned}
\widehat{Y}_r = &\sum_{j\in\mathcal{N}_r} \widetilde{\lambda}_{Y_r j}(\widehat{\mathcal{A}}_j + \widehat{\mu}_j) - \sum_{h\in\mathcal{H}_r} \widetilde{\Lambda}_{Y_r h}\widehat{\Lambda}_{Y_r h} + \sum_{h\in\mathcal{H}_r} \widetilde{\Lambda}_{Y_r h}\widehat{L}_h + \sum_{i\notin\mathcal{N}_r} (\Omega_{Y_r i} + \widetilde{\Lambda}_{Y_r i})(\widehat{q}_{ri} - \widehat{\Omega}_{Y_r i}) \\
&+ (1 - \sum_{i\notin\mathcal{N}_r} \Omega_{Y_r i} - \sum_{h\in\mathcal{H}_r} \widetilde{\Lambda}_{Y_r h} - \sum_{i\notin\mathcal{N}_r}\sum_{j\in\mathcal{N}_r} \widetilde{\lambda}_{Y_r j}\widetilde{\Omega}^{x}_{ji})\widehat{GDP}_r.
\end{aligned}
$$

Notice that

$$
\begin{aligned}
&1-\sum_{i\notin\mathcal{N}_r}\Omega_{Y_ri}-\sum_{i\notin\mathcal{N}_r}\sum_{j\in\mathcal{N}_r}\widetilde{\lambda}_{Y_rj}\widetilde{\Omega}^x_{ji}-\sum_{h\in\mathcal{H}_r}\widetilde{\Lambda}_{Y_rh}\\
=&1-\sum_{i\notin\mathcal{N}_r}\Omega_{Y_ri}-\sum_{i\in\mathcal{N}_r}\Omega_{Y_ri}\sum_{j\in\mathcal{N}_r}\widetilde{\psi}^{x_r}_{ij}(\sum_{h\in\mathcal{H}_r}\widetilde{\Omega}^{\ell}_{jh}+\sum_{i\notin\mathcal{N}_r}\widetilde{\Omega}^x_{ji})\\
=&1-\sum_{i\notin\mathcal{N}_r}\Omega_{Y_ri}-\sum_{i\in\mathcal{N}_r}\Omega_{Y_ri}\sum_{j\in\mathcal{N}_r}\widetilde{\psi}^{x_r}_{ij}(1-\sum_{i\in\mathcal{N}_r}\widetilde{\Omega}^x_{ji})\\
=&1-\sum_{i\in\mathcal{N}_r}\Omega_{Y_ri}-\sum_{i\notin\mathcal{N}_r}\Omega_{Y_ri}\\
=&1-\frac{1}{GDP_r}(\sum_{i\in\mathcal{N}_r}(S_i-\sum_{j\in\mathcal{N}_r}p_ix_{ji})-\sum_{i\notin\mathcal{N}_r}\sum_{j\in\mathcal{N}_r}p_ix_{ji})\\
=&1-\frac{1}{GDP_r}\sum_{i\in\mathcal{N}_r}(S_i-\sum_{j\in\mathcal{N}}p_jx_{ij})=0,
\end{aligned}
$$

where the third equality $(I_{N_r}-\widetilde{\Omega}^{D_r}_x)^{-1}=I_{N_r}+(I_{N_r}-\widetilde{\Omega}^{D_r}_x)^{-1}\widetilde{\Omega}^{D_r}_x$.

Finally, define the share of imports from foreign firm i relative to domestic GDP as $\Lambda_{Y_ri}=-\frac{p_iq_{ri}}{GDP_r}$ for $i\in\mathcal{N}-\mathcal{N}_r$

$$
\widehat{Y}_r=\sum_{j\in\mathcal{N}_r}\widetilde{\lambda}_{Y_rj}(\widehat{\mathcal{A}}_j+\widehat{\mu}_j)-\sum_{h\in\mathcal{H}_r}\widetilde{\Lambda}_{Y_rh}\widehat{\Lambda}_{Y_rh}+\sum_{h\in\mathcal{H}_r}\widetilde{\Lambda}_{Y_rh}\widehat{L}_h+\sum_{i\notin\mathcal{N}_r}(\widetilde{\Lambda}_{Y_ri}-\Lambda_{Y_ri})(\widehat{q}_{ri}-\widehat{\Lambda}_{Y_ri}).
$$