International Misallocation and Comovement under Production Networks

Alejandro Rojas-Bernal*

Vancouver School of Economics, The University of British Columbia

CLICK HERE FOR THE LATEST VERSION

October 9, 2023

Abstract

In this paper, I develop a general aggregation theory that explains the role of production networks in country-level TFP. This theory applies to a distorted production network open economy with endogenous factor supply. My main contribution is to provide decompositions for the country-level TFP variation that account for the possibility that factors of production and dividends cross national boundaries. The country-level TFP depends on sufficient statistics that characterize the effect on domestic real GDP from (i) firm-level productivity and markdown shocks in domestic and foreign firms and (ii) variations in the global income distribution. These decompositions do not require quantity measures of variations, facilitating their empirical implementation, as price data is no longer necessary. Additionally, for an efficient economy, a Hulten type of decomposition exists for each country, and the global sales distribution is a sufficient statistic to characterize the first-order propagation of global shocks on country-level TFP. These results support a theory of economic spillovers and contagion through industrial networks, corroborating the essential role of global value chains in creating strong complementarities and commonalities in business cycles across countries.

*Email: alejandrorojasecon@gmail.com

1 Introduction

Throughout much of human history, the landscape of production was fundamentally local. Firms leveraged local resources, and consumers predominantly sourced their products locally. Long-distance trade was primarily for luxury goods like precious metals, spices, and textiles. Prominent cities and civilizations harnessed these markets, amassing financial surpluses that allowed them to establish economic hegemony. For instance, mastery of seafaring techniques empowered the Phoenicians, bolstering their trade in cedar and linens, and the secrecy of sericulture gave total control to China over the global silk market. The onset of globalization, mainly after the Second World War, has altered the complexity and interconnectedness of markets. Today, consumers navigate an economy with intricate global supply chains, where the production of goods often extends across multiple continents. Moreover, financial globalization and digital advancements have reshaped production, creating dependencies on international factors at each stage of the supply chain.

This paper provides a novel country-level aggregate efficiency wedge decomposition that accounts for the possibility that factors of production and dividends cross national boundaries. The main contribution is to provide decompositions and sufficient statistics for the country-level total factor productivity (TFP) that account for the complexity of global supply chains and factoral reallocation across countries. My findings illustrate the interplay of global supply chains and economic spillovers through intermediate input markets.

The neoclassic framework that I use allows for (i) general production networks, (ii) sectoral rebated distortions, (iii) household heterogeneity in income and consumption across and within countries, (iv) fixed or endogenous factor supply, and (v) cross-country factoral markets.

I leverage the results from Rojas-Bernal (2023) that characterize the aggregate and idiosyncratic efficiencies wedges for a closed economy setting. In that paper, I obtained the decomposition for aggregate TFP in a general production network environment with distortions and household heterogeneity on consumption and income. In this paper, I extend the model to an open economy setting. This extension allows me to focus instead on country-level TFP. My model defines *country* as a collection of firms and households that produce and reside in a specific geographic space. Here, *country*-level TFP represents the aggregate efficiency for the collection of firms operating within that geography. In this sense, the *country*-level TFP decompositions obtained in this paper apply to any geographic space, e.g., regions, states, and cities. The crucial point of this research is leaving aside the assumption of country-specific factor markets. This grants flexibility to the decompositions and sufficient statistics, enabling their application across multiple economic environments.

I begin by obtaining sufficient statistics that characterize, for a specific country, the value-added contribution that any firm or factor in the global economy has. In an environment free of distortions or without intermediate inputs, a country's GDP is purely a product of domestic firms and the factors they employ. In other words, a country can only extract value added by utilizing factors that generate domestic production. With the introduction of distortions and input-output networks, countries can capitalize on intermediate inputs, producing domestic goods that yield surplus profits. In this way, foreign production and factors foreign firms use can directly contribute value-added to a country's GDP.

Using these statistics, I break down the first-order approximation of country-level TFP into three distinct channels. First is a direct technological effect. Second is the direct effect of variations in distortions. These two channels capture the propagation effects of productivity and distortion shocks for a country, keeping its share of global GDP and the factoral income distribution fixed. For these channels, shocks to domestic firms directly impact a country's TFP, while shocks to suppliers of intermediate inputs can indirectly affect TFP by causing profit fluctuations. Finally, a country's TFP increases with its global GDP share and with changes in the global factoral income distribution that make more affordable inputs essential for domestic production.

For a global economy without distortions, keeping the distributional channel fixed, country-level TFP increases with (i) higher domestic productivity, (ii) stronger distortions for foreign firms, and (iii) weaker distortions for domestic firms. Moreover, through the distributional channel, country-level TFP can increase with the labor income share for domestic factors as long as they have a small value-added on domestic production. In other words, making more expensive domestic factors is beneficial for a country if the value added by those factors reaches mainly foreign production.

For a global economy without distortions, with country-specific factor markets, and with complete equity home bias, there is Hulten (1978) theorem type of result that characterizes the envelope condition for the country's efficiency wedge. The variation in country-level TFP is solely driven by domestic productivity shocks, with the domestic sales distribution serving as a sufficient statistic for its first-order variation. Thus, only under these stringent constraints can one overlook network intricacies when assessing the aggregate impact of microeconomic shocks on country-level TFP.

The quantitative implementation applies the open economy Hulten's theorem to the long-run world input-output database. My decompositions highly correlate with observable and independent measures of country-level TFP.

Related Literature

This paper contributes to the literature on production networks, growth accounting, and misallocation. First, the research on shock propagation in production networks builds on the canonical multisector models from Hulten (1978) and Long & Plosser (1983). These models have been used to study the propagation of sectoral productivity shocks (Foerster et al., 2011; Horvath, 1998, 2000; Dupor, 1999; Acemoglu et al., 2012, 2016; Carvalho et al., 2021) and distortions (Basu, 1995; Ciccone, 2002; Yi, 2003; Jones, 2011, 2013; Asker et al., 2014; Baqaee, 2018; Liu, 2019; Baqaee & Farhi, 2020; Bigio & La'O, 2020; Rojas-Bernal, 2023). Huo et al. (2021) and Baqaee & Farhi (2023) implement these models in an open economy setting to study the comovement and propagation of shocks through global supply chains. The decompositions for country-level real GDP from Huo et al. (2021) apply for a CES economy where factors are sector-specific. The factor supply is also elastic, and distortions are wasted (iceberg costs). The country-level real GDP decompositions from Baqaee & Farhi (2023) apply to a general CRS production network economy where factoral markets are country-specific and distortions are rebated back to domestic households. This paper introduces the first decomposition for country-level real GDP and TFP in a general CRS production network economy with flexible factor markets, i.e., not sector- or country-specific. This flexibility allows for factoral reallocation effects within and across countries. The factor supply is elastic, and distortions are rebated back to households.

In the growth accounting literature opened by Solow (1957), and developed by Domar (1961); Hulten (1978); Jorgenson et al. (1987); Hall & Diamond (1990); Basu & Fernald (2002); Petrin & Levinsohn (2012); Osotimehin (2019); Baqaee & Farhi (2020); Rojas-Bernal (2023), I obtain a novel decomposition for country-level TFP that captures the cross country spillovers from productivities, distortions, and income distribution variations. The country-level decomposition of the effects from the reallocation of resources relates my model with the misallocation literature (Restuccia & Rogerson, 2008; Hsieh & Klenow, 2009).

Layout

The structure of the paper is as follows. Section 2 introduces the open economy multisector input-output model with heterogenous households and distortions. Section 3 characterizes the equilibrium and the network centrality measures. This section introduces new country-specific sufficient statistics that capture how firms and factors in the global economy affect the value-added distribution from a country. Section 4 presents the novel decompositions for country-level TFP and characterizes the sufficient statistics that are necessary for this representation. Section 5 explains how the results of this paper differ from Baqaee & Farhi (2023); in particular, I show that their results are a limiting case of my decomposition. Section 6 implements the open economy Hulten theorem derived in this paper using the world input-output database. Section 7 concludes.

2 General Framework

In this section, I set up a static nonparametric general equilibrium model with constant-returns-to-scale (CRS) for economies with N sectors and H types of households. Sector $i \in \mathscr{N} = \{1, \cdots, N\}$ consists of two types of firms: (i) a unit mass of monopolistic competitive firms indexed by $z_i \in [0,1]$ producing differentiated goods, and (ii) a perfectly competitive producer that aggregates the industry's differentiated goods into a uniform sectoral good that can be consumed by households or used by other firms as intermediate inputs. Firms differ along four dimensions; first, firms in sector $i \in \mathscr{N}_r \subseteq \mathscr{N}$ produce in the country r; second, monopolistic firms across sectors operate under different technologies; third, monopolistic firms within sectors have heterogeneous input demand; and fourth, sectoral aggregators face different distortions. Households of type $h \in \mathscr{H} = \{1, \cdots, H\}$ consume sectoral goods using the income received from their endogenous labor supply and rebated profits. Households differ along four dimensions; first, households of type $h \in \mathscr{H}_r \subseteq \mathscr{H}$ reside in the country r; second, their preferences; third, a type-specific horizontally differentiated labor supply; and fourth, the composition of their equity portfolio. Financial markets are incomplete, and households cannot cross-insure their idiosyncratic income shocks.

2.1 Production

Monopolistic firms within sectors produce differentiated goods using the same technology. The production for firm z_i in sector i follows

$$y_{z_i} = A_i \, Q_i \left(L_{z_i}, X_{z_i}\right), \quad L_{z_i} = A_i^\ell \, Q_i^\ell \left(\left\{ A_{ih}^\ell \, \ell_{z_i h} \right\}_{h \in \mathscr{H}} \right), \quad X_{z_i} = A_i^x \, Q_i^x \left(\left\{ A_{ij}^x \, x_{z_i j} \right\}_{j \in \mathscr{N}} \right), \quad (1)$$

where y_{z_i} stands for output, A_i is the sector-specific Hicks-neutral productivity term. L_{z_i} is the labor composite that depends on the productivity A_i^ℓ. $\ell_{z_i h}$ is the amount of labor hired from household h and is influenced by the productivity A_{ih}^ℓ. X_{z_i} is the intermediate input composite that depends on the productivity A_i^x. $x_{z_i j}$ is the amount of intermediate input goods purchased from sector j and is influenced by the productivity A_{ij}^x.

The technologies $Q_i : \mathbb{R}_+^2 \to \mathbb{R}_+$, $Q_i^\ell : \mathbb{R}_+^H \to \mathbb{R}_+$, and $Q_i^x : \mathbb{R}_+^N \to \mathbb{R}_+$ are neoclassical and satisfy the following regularity conditions: they are positive, finite, and for the set of labor types and intermediate inputs for which there is effective demand, they are monotonically increasing, twice continuously differentiable, strictly concave, and the Inada conditions hold.

The profits for firms z_i are given by

$$\pi_{z_i} = p_{z_i} y_{z_i} - \underbrace{\sum_{h \in \mathscr{H}} w_h \, \ell_{z_i h}}_{= p^{\ell}_{z_i} L_{z_i}} - \underbrace{\sum_{j \in \mathscr{N}} p_j \, x_{z_i j}}_{= p^{x}_{z_i} X_{z_i}}, \tag{2}$$

where p_{z_i} is the price of its output, $p^{\ell}_{z_i}$ is the price for the labor composite, $p^{x}_{z_i}$ is the price for the intermediate input composite, w_h is the wage received by households of type h, and p_j is the market price for the good produced by the competitive aggregator in sector j.

The competitive firm in sector i guarantees a homogeneous good by aggregating sectoral production using the following CES production function

$$y_i = \left(\int {y_{z_i}}^{\mu_i} \, dz_i \right)^{\frac{1}{\mu_i}}, \tag{3}$$

where $\mu_i \leq 1$ stands for the sector-specific markdown, and y_{z_i} represents the demand of goods produced by firm z_i. The aggregator takes prices as given and maximizes profits given by $\bar{\pi}_i = p_i y_i - \int p_{z_i} y_{z_i} \, dz_i$.

2.2 Households

Households of type h share the preference utility function $U_h\left(C_h, L_h\right)$, where C_h stands for real consumption, and L_h for the labor supply. The utility $U_h : \mathbb{R}^2_+ \to \mathbb{R}_+$ satisfies the usual regularity conditions: $U_{C_h} > 0$, $U_{L_h} < 0$, twice continuously differentiable, strictly concave, and the Inada conditions hold. The composite real consumption $C_h = Q^c_h\left(\{C_{hi}\}_{i \in \mathscr{N}}\right)$ depends on the final consumption C_{hi} of goods from sector i. The consumption aggregation technology $Q^c_h : \mathbb{R}^N_+ \to \mathbb{R}_+$ is neoclassical: positive, finite, homogeneous of degree one, and for the set of goods for which there is effective final demand, it is monotonically increasing, twice continuously differentiable, strictly concave, and the Inada conditions hold.

Each household is infinitesimal, and for this reason, they take prices and wages as given. Consequently, for any two households with type h, their choices are equivalent, and the notation of the model becomes simpler by assuming a type-specific representative household with a budget constraint given by

$$E_h = p^c_h C_h = \sum_{i \in \mathscr{N}} p_i C_{hi} \leq J_h + \Pi_h, \qquad \text{and} \qquad \Pi_h = \sum_{i \in \mathscr{N}} \kappa_{ih} \left(\bar{\pi}_i + \int \pi_{z_i} \, dz_i \right). \tag{4}$$

Expenditure E_h must not be greater than income; the latter includes labor income $J_h = w_h L_h$, and dividend income Π_h. Households of type h own a fraction κ_{ih} of the firms in sector i.

2.3 Market Clearing

For this economy, the technologies, productivities, markdowns, and ownership distributions are primitives. Monopolistic competition is the only source of market imperfections. These distortions reallocate resources and imply no wasted resources. Hence, the goods market clearing is given by

$$y_i = \sum_{h \in \mathscr{H}} C_{hi} + \sum_{j \in \mathscr{N}} x_{ji} \quad \forall i \in \mathscr{N}, \tag{5}$$

where $x_{ji} \equiv \int x_{z_j i}\, dz_j$ is the total amount of intermediate inputs from sector i bought by all monopolistic firms in sector j. Labor market clearing requires $L_h = \ell_h \ \forall h \in \mathscr{H}$, with $\ell_h = \sum_{i \in \mathscr{N}} \int \ell_{z_i h}\, dz_i$.

2.4 Remarks

This environment also applies to the following three generalizations. First, following McKenzie (1959), economies with variable (increasing or decreasing) return to scale can be handled by appropriately introducing producer-specific fixed entrepreneurial factors in a constant return model. Second, without loss of generality, the model and the following results apply to any production factor, not only labor. Finally, the effect of markdowns in the results from my model is isomorphic to other distortions that deviate the system of prices from its first-best solution, such as taxes and financial constraints.

A potential limitation of my model is that I assume segmentation of the labor supply across types of households. The parsimony from this premise allows me to bypass three problems. First, I do not need to consider an ownership matrix that specifies the factor share supplied by each household type. Second, I do not need to consider the cross-elasticities in preferences that arise from the supply of multiple factors by the same household. Third, I can abstract from strategic complementarities between multiple types of households in the supply of the same factor.

3 Equilibrium and Network Centralities

In this section, first, I characterize the equilibrium for this economy. Second, I introduce measures of bilateral centrality across firms and households, and measures of aggregate centrality that portray each firm or household's role in the economy. This section is essential to understand the first-order approximations that make up the main contribution of this paper.

3.1 Equilibrium Characterization

Let $e \equiv (\mathscr{A}, \mu, \kappa)$ represent the aggregate state, and $\mathscr{E}$ denote the measurable collection of all possible realizations for this state. The matrix $\mathscr{A} \equiv (A, A_\ell, A_x, \underline{A}_\ell, \underline{A}_x)$ collects all productivity measures,[1] and sectoral markdowns are captured by $\mu \equiv (\mu_1, \cdots, \mu_N)'$. The equity matrix $\kappa \equiv (\kappa_1, \cdots, \kappa_N)'$ of size $N \times H$ contains the ownership distribution of firms in sector i represented by the vector $\kappa_i \equiv (\kappa_{i1}, \cdots, \kappa_{iH})'$, with $\kappa_i' \mathbb{1}_H = 1$, and where $\mathbb{1}_H$ is an H sized vector of ones.

For this economy, a mapping of the realization of the aggregate state to an allocation $\vartheta = (\vartheta(e))_{e \in \mathscr{E}}$ and the price system $\rho = (\rho(e))_{e \in \mathscr{E}}$ is represented by the set of functions

$$\vartheta(e) \equiv \left\{ \left\{ \left(y_{z_i}(e), \{\ell_{z_i h}(e)\}_{h \in \mathscr{H}}, \{x_{z_i j}(e)\}_{j \in \mathscr{N}} \right)_{z_i \in [0,1]}, y_i(e), \{C_{hi}(e)\}_{h \in \mathscr{H}} \right\}_{i \in \mathscr{N}}, \{C_h(e), L_h(e)\}_{h \in \mathscr{H}} \right\},$$

$$\rho(e) \equiv \left\{ \left\{ \left(p_{z_i}(e), p^\ell_{z_i}(e), p^x_{z_i}(e) \right)_{z_i \in [0,1]}, p_i(e) \right\}_{i \in \mathscr{N}}, \{w_h(e), p^c_h(e)\}_{h \in \mathscr{H}} \right\}.$$

To make the notation cleaner, the definitions and implementation of the model that follows are conditional in a specific aggregate state $e \in \mathscr{E}$, e.g., $\mu(e)$ is portrayed by μ.

Definition 1. For any realization of the aggregate state e in the state space $\mathscr{E}$, an equilibrium is the combination of an allocation and a price system (ϑ, ρ) such that:

(i) given wages $\{w_h\}_{h \in \mathscr{H}}$ and prices $\{p_j\}_{j \in \mathscr{N}}$, monopolistically competitive firms' labor $\{\ell_{z_i h}\}_{h \in \mathscr{H}}$ and intermediate input demand $\{x_{z_i j}\}_{j \in \mathscr{N}}$, output y_{z_i}, and price p_{z_i} maximize their profits;

(ii) given prices $[p_{z_i}]_{z_i \in [0,1]}$, aggregator firms' good demand $[y_{z_i}]_{z_i \in [0,1]}$, and output y_i maximize their profits;

(iii) given prices $\{p_i\}_{i \in \mathscr{N}}$ and wages $\{w_h\}_{h \in \mathscr{H}}$, households' consumption bundles $\{C_{hi}\}_{i \in \mathscr{N}}$ and labor supply L_h maximize utility while satisfying their budget constraint;

(iv) goods and labor markets clear.

I will abstract from within sector firm heterogeneity by imposing the assumption of symmetry, i.e., $\ell_{ih} = \ell_{z_i h}$, and $x_{ij} = x_{z_i j}$ $\forall z_i \in [0,1]$, $\forall i, j \in \mathscr{N}$ and $\forall h \in \mathscr{H}$.[2] For this reason, I will refer indistinguishably to firm z_i as firm i.

Proposition 1. The set of functions (ϑ, ρ) are an equilibrium if and only if the following set

[1] $A \equiv (A_1, \cdots, A_N)'$, $A_\ell = \left(A^\ell_1, \cdots, A^\ell_N\right)'$, $A_x \equiv (A^x_1, \cdots, A^x_N)'$, $\underline{A}_\ell = \left(\underline{A}^\ell_1, \cdots, \underline{A}^\ell_N\right)'$, $\underline{A}_x = (\underline{A}^x_1, \cdots, \underline{A}^x_N)'$, $\underline{A}^\ell_i = \left(A^\ell_{i1}, \cdots, A^\ell_{iH}\right)'$, and $\underline{A}^x_i = (A^x_{i1}, \cdots, A^x_{iN})'$.

[2] As a consequence $y_i = y_{z_i}$, $p_i = p_{z_i}$, $L_i = L_{z_i}$, and $X_i = X_{z_i}$.

of conditions are jointly satisfied $\forall e \in \mathscr{E}$

$$\frac{\partial C_h / \partial C_{hj}}{\partial C_h / \partial C_{hi}} = \mu_i \frac{\partial y_{z_i}}{\partial x_{z_i j}} \quad \forall i, j \in \mathscr{N}, \text{ and } \forall h \in \mathscr{H} \text{ such that } C_{hi} > 0,\ C_{hj} > 0, \text{ and } x_{z_i j} > 0, \tag{6}$$

$$-\frac{w_b}{w_h}\frac{U_{L_h}}{U_{C_{hi}}} = \mu_i \frac{\partial y_i}{\partial \ell_{ib}} \quad \forall i \in \mathscr{N}, \text{ and } \forall h, b \in \mathscr{H} \text{ such that } C_{hi} > 0, \text{ and } \ell_{ib} > 0, \tag{7}$$

and resource constraints

$$\begin{aligned} y_i &= \sum_{h \in \mathscr{H}} C_{hi} + \sum_{j \in \mathscr{N}} x_{ji} \quad \forall i \in \mathscr{N}, \\ \text{and} \qquad L_h &= \sum_{i \in \mathscr{N}} \ell_{ih} \quad \forall h \in \mathscr{H}. \end{aligned} \tag{8}$$

Proposition 1 identifies the set of equilibrium allocations. In equation (6), for firm i, the markdown-adjusted marginal productivity from using the good from sector j as an intermediate input has to equate for every household the marginal rate of substitution between goods i and j. In equation (7), for firm i, the markdown-adjusted marginal productivity from using the labor supplied by households of type b, has the equate for every household a wage-adjusted marginal rate of substitution between the consumption of the good from sector i and their labor supply. This equilibrium is the same as in Rojas-Bernal (2023).

3.2 Measures of Centrality

For the following measures, downstream or cost centrality refers to the propagation of costs from the supply of labor or intermediate inputs through supply chains, and upstream or revenue centrality refers to the propagation of money flows from the demand for labor and goods through payment chains. Table 1 summarizes the direct centralities and Table 2 the network centralities.

3.2.1 Direct Centralities

The vectors $\omega_\ell \equiv \left(\omega_1^\ell, \cdots, \omega_N^\ell\right)'$ and $\omega_x \equiv \left(\omega_1^x, \cdots, \omega_N^x\right)'$ portray the direct cost centralities from composites. Its elements $\omega_i^\ell \equiv \frac{\partial \log c_i(\vartheta,\rho)}{\partial \log p_i^\ell} = \frac{p_i^\ell L_i}{c_i(\vartheta,\rho)}$ and $\omega_i^x \equiv \frac{\partial \log c_i(\vartheta,\rho)}{\partial \log p_i^x} = \frac{p_i^x X_i}{c_i(\vartheta,\rho)}$ capture respectively firm i's cost elasticities to p_i^ℓ and p_i^x, and in equilibrium they equal the cost share of the labor and intermediate input composites. For this reason, $\omega_i^\ell + \omega_i^x = 1$.

The matrices $\widetilde{\Omega}_\ell$ and $\widetilde{\Omega}_x$ depict direct labor and intermediate input downstream centralities. Its elements $\widetilde{\Omega}_{ih}^\ell \equiv \frac{\partial \log c_i(\vartheta,\rho)}{\partial \log w_h} = \frac{w_h \ell_{ih}}{c_i(\vartheta,\rho)}$ and $\widetilde{\Omega}_{ij}^x \equiv \frac{\partial \log c_i(\vartheta,\rho)}{\partial \log p_j} = \frac{p_j x_{ij}}{c_i(\vartheta,\rho)}$ capture respectively firm i's cost elasticities to w_h and p_j, and in equilibrium they equal the cost share of the labor supplied by households of type h and the good from firm j. The fact that $\sum_{h\in\mathscr{H}} \widetilde{\Omega}_{ih}^\ell + \sum_{j\in\mathscr{N}} \widetilde{\Omega}_{ij}^x = 1$ indicate that all costs come from labor or intermediate inputs.

Table 1: Direct Centralities

Matrix	***Definition***	***In Equilibrium***	***Properties***
ω_ℓ	$\omega_i^\ell \equiv \frac{\partial \, log \, c_i(\vartheta,\rho)}{\partial \, log \, p_i^\ell}$	Cost share of L_i	$\omega_i^\ell + \omega_i^x = 1$
ω_x	$\omega_i^x \equiv \frac{\partial \, log \, c_i(\vartheta,\rho)}{\partial \, log \, p_i^x}$	Cost share of X_i	
$\widetilde{\Omega}_\ell$	$\widetilde{\Omega}_{ih}^\ell \equiv \frac{\partial \, log \, c_i(\vartheta,\rho)}{\partial \, log \, w_h}$	Cost share of ℓ_{ih}	$\sum_{h \in \mathscr{H}} \widetilde{\Omega}_{ih}^\ell + \sum_{j \in \mathscr{N}} \widetilde{\Omega}_{ij}^x = 1$
$\widetilde{\Omega}_x$	$\widetilde{\Omega}_{ij}^x \equiv \frac{\partial \, log \, c_i(\vartheta,\rho)}{\partial \, log \, p_j}$	Cost share of x_{ij}	
$diag\,(\omega_\ell)\,\alpha = \widetilde{\Omega}_\ell$	$\alpha_{ih} \equiv \frac{\partial \, log \, p_i^\ell \, L_i}{\partial \, log \, w_h}$	Cost share of ℓ_{ih} in L_i	$\sum_{h \in \mathscr{H}} \alpha_{ih} = 1$
$diag\,(\omega_x)\,\mathscr{W} = \widetilde{\Omega}_x$	$\omega_{ij} \equiv \frac{\partial \, log \, p_i^x \, X_i}{\partial \, log \, p_j}$	Cost share of x_{ij} in X_i	$\sum_{j \in \mathscr{N}} \omega_{ij} = 1$
β	$\beta_{hi} \equiv \frac{\partial \, log \, E_h}{\partial \, log \, p_i}$	Cost share of C_{hi}	$\sum_{i \in \mathscr{N}} \beta_{hi} = 1$
κ	$\kappa_{ih} \equiv \frac{d \, \Pi_h}{d \, \pi_i}$	Equity share of h in i	$\sum_{h \in \mathscr{H}} \kappa_{ih} = 1$
$\Omega_\ell \equiv diag\,(\mu)\,\widetilde{\Omega}_\ell$	$\Omega_{ih}^\ell \equiv \frac{\partial \, log \, S_i}{\partial \, log \, w_h}$	Share of S_i for ℓ_{ih}	
$\Omega_x \equiv diag\,(\mu)\,\widetilde{\Omega}_x$	$\Omega_{ij}^x \equiv \frac{\partial \, log \, S_i}{\partial \, log \, p_j}$	Share of S_i for x_{ij}	$\sum_{h \in \mathscr{H}} \left(\Omega_{ih}^\ell + \Omega_{ih}^\pi\right) + \sum_{j \in \mathscr{N}} \Omega_{ij}^x = 1$
$\Omega_\pi = diag\,(\mathbb{1}_N - \mu)\,\kappa$	$\Omega_{ih}^\pi = \frac{\kappa_{ih} \, \pi_i}{S_i}$	Share of S_i for Π_h	

Using these definitions, I obtain the labor network $\alpha \equiv diag\,(\omega_\ell)^{-1}\,\widetilde{\Omega}_\ell$ and the input-output network $\mathscr{W} \equiv diag\,(\omega_x)^{-1}\,\widetilde{\Omega}_x$, where $diag$ stands for the diagonal operator. Its elements $\alpha_{ih} \equiv \frac{\partial \, log \, p_i^\ell \, L_i}{\partial \, log \, w_h} = \frac{w_h \, \ell_{ih}}{p_i^\ell \, L_i}$ and $\omega_{ij} \equiv \frac{\partial \, log \, p_i^x \, X_i}{\partial \, log \, p_j} = \frac{p_j \, x_{ij}}{p_i^x \, X_i}$ capture respectively firm i's composite cost elasticities to w_h and p_j, and in equilibrium they equal the corresponding composites' cost share of the labor supplied by households of type h and the good from firm j. Notice that $\sum_{h \in \mathscr{H}} \alpha_{ih} = 1$ and $\sum_{j \in \mathscr{N}} \omega_{ij} = 1$.

From here, I can define the revenue-based upstream centrality matrices $\Omega_\ell \equiv diag\,(\mu)\,\widetilde{\Omega}_\ell$ and $\Omega_x \equiv diag\,(\mu)\,\widetilde{\Omega}_x$. Since $\mu_i \in (0,1] \; \forall i \in \mathscr{N}$, $\widetilde{\Omega}_\ell \succcurlyeq \Omega_\ell$ and $\widetilde{\Omega}_x \succcurlyeq \Omega_x$, where $\succcurlyeq$ stands for elementwise greater than or equal to. Its elements $\Omega_{ih}^\ell \equiv \frac{\partial \, log \, S_i}{\partial \, log \, w_h} = \frac{w_h \, \ell_{ih}}{S_i}$ and $\Omega_{ij}^x \equiv \frac{\partial \, log \, S_i}{\partial \, log \, p_j} = \frac{p_j \, x_{ij}}{S_i}$ capture respectively the elasticities of firm i's sales to w_h and p_j, and in equilibrium they equal the sales share of payments for labor supplied by workers of type h and goods from firm j. Additionally, $\Omega_{ih}^\pi = \frac{\kappa_{ih} \, \pi_i}{S_i}$ portrays the equilibrium sales share of firm i's profits rebated back to households of type h. The fact that $\sum_{h \in \mathscr{H}} \Omega_{ih}^\ell + \sum_{j \in \mathscr{N}} \Omega_{ij}^x + \sum_{b \in \mathscr{H}} \Omega_{ib}^\pi = 1$ indicate that all revenue generated by firm i ends as payments for labor, intermediate inputs, or dividends.

Finally, for households, the consumption network $\beta = (\beta_1, \cdots, \beta_H)'$ contains the vectors $\beta_h \equiv (\beta_{h1}, \cdots, \beta_{hN})'$. Its element $\beta_{hi} \equiv \frac{\partial \, log \, E_h}{\partial \, log \, p_i} = \frac{p_i \, C_{hi}}{E_h}$ captures the expenditure elasticity for households of type h to p_i, and in equilibrium they equal the expenditure share on the good supplied by firm i. For this reason $\sum_{i\in\mathscr{N}}\beta_{hi}=1$.

3.2.2 Network Adjusted Centralities

The firm-to-firm downstream centrality matrix or cost-based Leontief inverse matrix is given by $\widetilde{\Psi}_x \equiv \left(I-\widetilde{\Omega}_x\right)^{-1} \equiv \sum_{q=0}^{\infty}\widetilde{\Omega}_x^q$. Its element $\widetilde{\psi}_{ij}^x$ captures the centrality of intermediate inputs supplied by firm j on the costs of firm i. Similarly, I define the firm-to-firm upstream centrality matrix or revenue-based Leontief inverse matrix $\Psi_x \equiv (I-\Omega_x)^{-1} \equiv \sum_{q=0}^{\infty}\Omega_x^q$, where its element ψ_{ij}^x represents the revenue share from firm i that through the payment of intermediate input reaches sales of firm j.

The firm-to-consumer downstream centrality matrix is given by $\widetilde{\mathscr{B}} \equiv \beta\,\widetilde{\Psi}_x$. Its element $\widetilde{\mathscr{B}}_{hi} = \sum_{j\in\mathscr{N}}\beta_{hj}\,\widetilde{\psi}_{ji}^x$ captures all direct or indirect paths through which the costs of firm i can reach the expenditure for households of type h. The cost-based sales Domar weight $\widetilde{\lambda}_i = \sum_{h\in\mathscr{H}}\chi_h\,\widetilde{\mathscr{B}}_{hi}$ stands for the average firm-to-consumer centrality from sector i, where $\chi_h = E_h/GDP$ represents the expenditure share for households of type h. Likewise, I define the consumer-to-firm upstream centrality matrix $\mathscr{B} \equiv \beta\,\Psi_x$, where its element $\mathscr{B}_{hi} = \sum_{j\in\mathscr{N}}\beta_j\,\psi_{ji}^x$ represents the share of expenditure from households of type h that through the payment chain reaches the revenue of firm i. The revenue-based sales Domar weight $\lambda_i = \sum_{h\in\mathscr{H}}\chi_h\,\mathscr{B}_{hi} = S_i/GDP$ stands for the average consumer-to-firm centrality towards sector i, and in equilibrium it coincides with the ratio of sales to GDP. These definitions generalize the supplier centrality vector from Baqaee (2018), or the influence vector from Acemoglu et al. (2012), to an environment with heterogeneous households and distortions.

The worker-to-firm downstream centrality matrix is given by $\widetilde{\Psi}_\ell \equiv \widetilde{\Psi}_x\,\widetilde{\Omega}_\ell$. Given that $\sum_{h\in\mathscr{H}}\widetilde{\psi}_{ih}^\ell = 1$, all costs for a firm can be traced back through the production network to some original labor cost. As a consequence, $\widetilde{\psi}_{ih}^\ell$ is the value-added share by workers of type h on the production process of firm i. In the same way, I define the firm-to-worker upstream centrality matrix $\Psi_\ell \equiv \Psi_x\,\Omega_\ell$, where the element ψ_{ih}^ℓ represents the revenue share from firm i that reaches labor income for workers of type h.

The worker-to-consumer downstream centrality matrix is given by $\widetilde{\mathscr{C}} \equiv \beta\,\widetilde{\Psi}_\ell$. Given that $\sum_{b\in\mathscr{H}}\widetilde{\mathscr{C}}_{hb} = 1$, its element $\widetilde{\mathscr{C}}_{hb}$ represents the value-added share for households of type h attributed to workers of type b. The cost-based factor Domar weight $\widetilde{\Lambda}_h = \sum_{b\in\mathscr{H}}\chi_b\,\widetilde{\mathscr{C}}_{bh}$ stands for the average worker-to-consumer centrality from workers of type h. Consequently, $\widetilde{\Lambda}_h$ is the share of aggregate value-added by their labor. All the costs from this economy originate in labor costs, and for this reason, $\sum_{h\in\mathscr{H}}\widetilde{\Lambda}_h = 1$. Similarly, the consumer-to-worker upstream centrality matrix is given by $\mathscr{C} \equiv \beta\,\Psi_\ell$, where its element $\mathscr{C}_{hb}$ portrays the share of consumption expenditure from households of type h that reaches labor income for workers of type b. The revenue-based factor Domar weight $\Lambda_h = \sum_{b\in\mathscr{H}}\chi_b\,\mathscr{C}_{bh} = J_h/GDP$ stands for the average

Table 2: **Network Adjusted Centralities**

Matrix	*Definition in Equilibrium*	*Properties*
Downstream or Cost-Based Centralities		
$\widetilde{\Psi}_x = \left(I - \widetilde{\Omega}_x\right)^{-1}$	$\widetilde{\psi}^x_{ij}$ *firm-to-firm* Centrality of j in the costs of i	
$\widetilde{\mathscr{B}} = \beta\, \widetilde{\Psi}_x$	$\widetilde{\mathscr{B}}_{hi}$ *firm-to-consumer* Centrality of i in the costs of h	
$\widetilde{\Psi}_\ell = \widetilde{\Psi}_x\, \widetilde{\Omega}_\ell$	$\widetilde{\psi}^\ell_{ih}$ *worker-to-firm* Value-added share by h in the production of i	$\sum_{h\in\mathscr{H}} \widetilde{\psi}^\ell_{ih} = 1$
$\widetilde{\mathscr{C}} = \beta\, \widetilde{\Psi}_\ell$	$\widetilde{\mathscr{C}}_{hb}$ *worker-to-consumer* Value-added share by b in the consumption of h	$\sum_{b\in\mathscr{H}} \widetilde{\mathscr{C}}_{hb} = 1$
$\widetilde{\lambda} = \widetilde{\mathscr{B}}'\, \chi$	$\widetilde{\lambda}_i$ *cost-based Domar weight* Share of aggregate value-added that passes through i	$\sum_{i\in\mathscr{N}} \omega^\ell_i\, \widetilde{\lambda}_i = 1$
$\widetilde{\Lambda} = \widetilde{\mathscr{C}}'\, \chi$	$\widetilde{\Lambda}_h$ *cost-based labor share* Share of aggregate value-added generated by h	$\sum_{h\in\mathscr{H}} \widetilde{\Lambda}_h = 1$
Upstream or Revenue-Based Centralities		
$\Psi_x = (I - \Omega_x)^{-1}$	ψ^x_{ij} *firm-to-firm* Share of S_i that reaches S_j	
$\mathscr{B} = \beta\, \Psi x$	$\mathscr{B}_{hi}$ *consumer-to-firm* Share of E_h that reaches S_i	
$\Psi_\ell = \Psi_x\, \Omega_\ell$	ψ^ℓ_{ih} *firm-to-worker* Share of S_i that reaches J_h	$\psi^\ell_i = \sum_{h\in\mathscr{H}} \psi^\ell_{ih}$
$\mathscr{C} = \beta\, \Psi_\ell$	$\mathscr{C}_{hb}$ *consumer-to-worker* Share of E_h that reaches J_h	$\mathscr{C}_h = \sum_{b\in\mathscr{H}} \mathscr{C}_{hb}$
$\lambda = \mathscr{B}'\, \chi$	λ_i *revenue-based Domar weight* Aggregate sales share S_i/GDP	$\sum_{i\in\mathscr{N}} \lambda_i \geq 1$
$\Lambda = \mathscr{C}'\, \chi$	Λ_h *revenue-based labor share* Labor income share J_h/GDP	$\Gamma = \sum_{h\in\mathscr{H}} \Lambda_h \leq 1$
$\chi = (\Omega_\ell + \Omega_\pi)'\, \lambda$	χ_h *expenditure share* Consumption expenditure share χ_h/GDP	$\sum_{h\in\mathscr{H}} \chi_h = 1$
Other Definitions		
$\delta = diag\,(\Lambda)^{-1}\, \Lambda$	δ_h *distortion centrality* Measure for how undervalue is L_h	$\delta_h = \widetilde{\Lambda}_h/\Lambda_h$
$M = \mathscr{C}\, \delta$	M_h *expenditure efficiency* Average distortion centrality faced by E_h	$M_h = \sum_{b\in\mathscr{H}} \mathscr{C}_{hb}\, \delta_b$
$F = \Psi_\ell\, \delta$	F_i *revenue efficiency* Average distortion centrality faced by S_i	$F_i = \sum_{h\in\mathscr{H}} \psi^\ell_{ih}\, \delta_h$

consumer-to-worker centrality towards workers of type h. In equilibrium, Λ_h coincides with the ratio of labor income to GDP.

Cost-based centralities are greater than or equal to revenue-based centralities, i.e., $\widetilde{\Psi}_x \succcurlyeq \Psi_x$, $\widetilde{\mathscr{B}} \succcurlyeq \mathscr{B}$, $\widetilde{\Psi}_\ell \succcurlyeq \Psi_\ell$, $\widetilde{\mathscr{C}} \succcurlyeq \mathscr{C}$, $\widetilde{\lambda} \succcurlyeq \lambda$, and $\widetilde{\Lambda} \succcurlyeq \Lambda$. For this reason, for workers of type h, $\delta_h = \widetilde{\Lambda}_h / \Lambda_h \geq 1$ is a measure of *distortion centrality* that captures how undervalued a worker is in the market. When workers supply their labor to sectors that operate in heavily distorted supply chains, their distortion centrality will be high, and a higher share of their value-added will reach households' income via rebated distortions. For this reason, $M_h = \sum_{b \in \mathscr{H}} \mathscr{C}_{hb}\, \delta_b$ and $F_i = \sum_{h \in \mathscr{H}} \psi^\ell_{ih}\, \delta_h$ capture the average distortion centrality faced by the consumption expenditure from households of type h and the revenue from firms in sector i. For M_h and F_i to be relatively high, it is necessary that the consumer-to-worker $\{\mathscr{C}_{hb}\}_{b \in \mathscr{H}}$ and the firm-to-worker $\left\{\psi^\ell_{ih}\right\}_{h \in \mathscr{H}}$ centralities are high, and this requires that the demand for goods and inputs is relatively undistorted. For this reason, M_h and F_i will be respectively called *expenditure efficiency* and *revenue efficiency*.

Additionally, for households of type h and firm i, I will respectively use $\mathscr{C}_h = \sum_{b \in \mathscr{H}} \mathscr{C}_{hb}$ and $\psi^\ell_i = \sum_{h \in \mathscr{H}} \psi^\ell_{ih}$ to capture their payment centrality, i.e., the share of their expenditure that reaches households' income via labor income. Notice that the cost-based equivalent for $\mathscr{C}_h$ and ψ^ℓ_i are equal to one, which implies that these measures will shrink as the influence from distortions rises.

Finally, in equilibrium, the expenditure shares are connected to the revenue-based Domar weights via the following relationship $\chi_h = \Lambda_h + \sum_{i \in \mathscr{N}} \Omega^\pi_{ih}\, \lambda_i$, and by definition $\sum_{h \in \mathscr{H}} \chi_h = 1$.

3.3 Gross Domestic Product and Gross National Income

Nominal GDP for country $r \in \mathscr{R}$ equals the revenue from domestic firms minus their intermediate input costs

$$GDP_r = \sum_{i \in \mathscr{N}_r} \left(1 - \sum_{j \in \mathscr{N}} \Omega^x_{ij} \right) S_i = \sum_{i \in \mathscr{N}_r} (1 - \mu_i\, \omega^x_i)\, S_i. \tag{9}$$

This definition coincides with the total value-added extracted by domestic firms, i.e., total labor costs and dividends

$$GDP_r = \sum_{i \in \mathscr{N}_r} \left(\sum_{h \in \mathscr{H}} w_h\, \ell_{ih} + (1 - \mu_i)\, S_i \right).$$

Gross National Income (GNI) is equal to the consumption expenditure from domestic households

$$GNI_r \equiv \sum_{h \in \mathscr{H}_r} E_h.$$

The redistribution of labor income and dividend income across countries generates country-level differences between GDP_r and GNI_r. Without distortions, due to balance trade, $GDP_r = GNI_r$. These differences cancel out at the global level, and the following relationship holds

$$GDP \equiv GNI \equiv \sum_{h \in \mathscr{H}} E_h.$$

The share of country r's GDP in the global economy is given by $\Phi_r = GDP_r/GDP$.

3.4 Value Added Extraction

The direct expenditure intensity from consumers h on final goods produced by country r is given by $\beta_{h|r} = \sum_{i \in \mathscr{N}_r} \beta_{hi}$. The direct expenditure intensity from firm i on intermediate inputs from country r is given by $\Omega_{i|r} = \sum_{j \in \mathscr{N}_r} \Omega^x_{ij}$.

The net network adjusted exposure of firm i to firms in sector j is given by

$$\widetilde{\psi}^x_{ij\,(\widetilde{\Omega}-\Omega)} = \sum_{m \in \mathscr{N}} \left(\widetilde{\Omega}^x_{im} - \Omega^x_{im} \right) \widetilde{\psi}^x_{mj} = (1 - \mu_i) \sum_{m \in \mathscr{N}} \widetilde{\Omega}^x_{im} \, \widetilde{\psi}^x_{mj}.$$

When firms in sector i operate competitively $\widetilde{\psi}^x_{ij\,(\widetilde{\Omega}-\Omega)} = 0$. However, firm i can charge a surplus over its intermediate input costs when it faces distortions. Each of the m intermediate input suppliers for firm i has an indirect cost exposure to firm j equal $\widetilde{\psi}^x_{mj}$. The intensity of the direct cost exposure from firm i to sector m is $\widetilde{\Omega}^x_{im}$, and firm i charges a $1 - \mu_i$ surplus over these costs.

For this reason,

$$\ddot{\lambda}^r_j = \mathbb{1}\left\{ j \in \mathscr{N}_r \right\} \frac{\lambda_j}{\Phi_r} + \sum_{i \in \mathscr{N}_r} \frac{\lambda_i}{\Phi_r} \widetilde{\psi}^x_{ij\,(\widetilde{\Omega}-\Omega)} = \mathbb{1}\left\{ j \in \mathscr{N}_r \right\} \frac{\lambda_j}{\Phi_r} + \sum_{i \in \mathscr{N}_r} (1 - \mu_i) \frac{\lambda_i}{\Phi_r} \sum_{m \in \mathscr{N}} \widetilde{\Omega}^x_{im} \, \widetilde{\psi}^x_{mj} \tag{10}$$

represents the share of value added in country r that can be traced back to the production from firms in sector j. Value added produced in sector j can be extracted by country r in two ways. First, by producing the goods domestically. Second, using intermediate inputs to produce domestic goods and charging a surplus distributed via dividends. For the global economy (G), it is the case that $\ddot{\lambda}^G_i = \widetilde{\lambda}_i$.

Country r's network multiplier is given by $\xi_r = \sum_{i \in \mathscr{N}} \ddot{\lambda}_i^r$. Without domestic intermediate input consumption $\xi_r = 1$. Without distortions but with some degree of domestic intermediate input consumption $\xi_r = \frac{\sum_{i \in \mathscr{N}_r} \lambda_i}{\sum_{i \in \mathscr{N}_r} \left(1 - \mu_i \, \omega_i^x\right) \lambda_i} \geq 0$. In general $\xi_r \geq 1$.

Similarly, the net network adjusted exposure of firm i to workers of type h is given by

$$\widetilde{\psi}^{\ell}_{ih\,(\widetilde{\Omega}-\Omega)} = \sum_{j \in \mathscr{N}} \left(\widetilde{\Omega}^x_{ij} - \Omega^x_{ij} \right) \widetilde{\psi}^{\ell}_{jh} = (1 - \mu_i) \sum_{j \in \mathscr{N}} \widetilde{\Omega}^x_{ij} \, \widetilde{\psi}^{\ell}_{jh}.$$

When firms in sector i operate competitively $\widetilde{\psi}^{\ell}_{ij\,(\widetilde{\Omega}-\Omega)} = 0$. However, firm i can charge a surplus over its intermediate input costs when it faces distortions. Each of the j intermediate input suppliers for firm i has an indirect cost exposure to workers of type h equal $\widetilde{\psi}^{\ell}_{jh}$. The intensity of the direct cost exposure from firm i to sector m is $\widetilde{\Omega}^x_{im}$, and firm i charges a $1 - \mu_i$ surplus over these costs.

For this reason,

$$\ddot{\Lambda}^r_h = \sum_{i \in \mathscr{N}_r} \frac{\lambda_i}{\Phi_r} \left(\widetilde{\Omega}^{\ell}_{ih} + \widetilde{\psi}^{\ell}_{ih\,(\widetilde{\Omega}-\Omega)} \right) = \sum_{i \in \mathscr{N}_r} \frac{\lambda_i}{\Phi_r} \left(\widetilde{\Omega}^{\ell}_{ih} + (1 - \mu_i) \sum_{j \in \mathscr{N}} \widetilde{\Omega}^x_{ij} \, \widetilde{\psi}^{\ell}_{jh} \right) \tag{11}$$

represents the share of value added in country r that can be traced back to the labor supply from workers of type h. Value added generated by workers of type h can be extracted by country r in two ways. First, by hiring them directly and producing goods. Second, procuring intermediate inputs that directly or indirectly require labor from h, using them to produce domestic goods, and charging a surplus distributed via dividends. Notice that $\left\{ \ddot{\Lambda}^r_h \right\}_{h \in \mathscr{H}}$ characterizes a distribution for sources of value-added for country r because $\sum_{h \in \mathscr{H}} \ddot{\Lambda}^r_h = 1$. For the global economy, it is the case that $\ddot{\Lambda}^G_h = \widetilde{\Lambda}_h$.

For workers of type h, $\delta^r_h = \ddot{\Lambda}^r_h / \Lambda_h$ represents the *distortion centrality* conditional on the value-added distribution for country r. From the perspective of country r, a worker is overvalued when $0 \leq \delta^r_h < 1$. $M^r_h = \sum_{b \in \mathscr{H}} \mathscr{C}_{hb} \, \delta^r_b$ and $F^r_i = \sum_{h \in \mathscr{H}} \psi^{\ell}_{ih} \, \delta^r_h$ capture the average distortion centrality faced by the consumption expenditure from household of type h and the revenue from firms in sector i conditional on the value-added distribution for country r. M^r_h and F^r_i will be respectively called *expenditure efficiency on country* r *revenue efficiency on country* r.

4 Open Economy Accounting

In this section, I derive the nonparametric ex-post sufficient statistics necessary to characterize the first-order variations in prices, labor wedges, labor income shares, and country-level TFP.

Table 3: Definitions Specific to Country r

Concept	***Definition***	***Properties***
Gross domestic product	$GDP_r = \sum_{i\in\mathscr{N}_r} (1-\mu_i\,\omega_i^x)\,\lambda_i$	$GDP = \sum_{r\in\mathscr{R}} GDP_r$
Gross national income	$GNI_r = \sum_{h\in\mathscr{H}_r} E_r$	$GNI = \sum_{r\in\mathscr{R}} GNI_r$
GDP share	$\Phi_r = \frac{GDP_r}{GDP}$	$\sum_{r\in\mathscr{R}} \Phi_r = 1$
Expenditure intensity from consumer h	$\beta_{h\vert r} = \sum_{i\in\mathscr{N}_r} \beta_{hi}$	$\sum_{r\in\mathscr{R}} \beta_{h\vert r} = 1$
Expenditure intensity from firm i	$\Omega_{i\vert r} = \sum_{j\in\mathscr{N}_r} \Omega_{ij}^x$	$\sum_{r\in\mathscr{R}} \Omega_{i\vert r} = \mu_i\,\omega_i^x$
Net network adjusted exposure of firm i to firms in sector j	$\widetilde{\psi}^x_{ij\,(\widetilde{\Omega}-\Omega)} = \sum_{m\in\mathscr{N}} \left(\widetilde{\Omega}^x_{im} - \Omega^x_{im}\right)\widetilde{\psi}^x_{mj}$	
Share of value added that can be traced back to sector j	$\ddot{\lambda}^r_j = \mathbb{1}\{j\in\mathscr{N}_r\}\frac{\lambda_j}{\Phi_r} + \sum_{i\in\mathscr{N}_r}\frac{\lambda_i}{\Phi_r}\widetilde{\psi}^x_{ij\,(\widetilde{\Omega}-\Omega)}$	
Network multiplier	$\xi_r = \sum_{i\in\mathscr{N}} \ddot{\lambda}^r_i$	$\xi_r \geq 1$
Net network adjusted exposure of firm i to workers of type h	$\widetilde{\psi}^\ell_{ih\,(\widetilde{\Omega}-\Omega)} = \sum_{j\in\mathscr{N}} \left(\widetilde{\Omega}^x_{ij} - \Omega^x_{ij}\right)\widetilde{\psi}^\ell_{jh}$	
Share of value added that can be traced back to h	$\ddot{\Lambda}^r_h = \sum_{i\in\mathscr{N}_r}\frac{\lambda_i}{\Phi_r}\left(\widetilde{\Omega}^\ell_{ih} + \widetilde{\psi}^\ell_{ih\,(\widetilde{\Omega}-\Omega)}\right)$	$\sum_{h\in\mathscr{H}} \ddot{\Lambda}^r_h = 1$
Distortion centrality for h	$\delta^r_h = \frac{\ddot{\Lambda}^r_h}{\Lambda_h}$	
Expenditure efficiency for h	$M^r_h = \sum_{b\in\mathscr{H}} \mathscr{C}_{hb}\,\delta^r_b$	
Expenditure efficiency for i	$F^r_i = \sum_{h\in\mathscr{H}} \psi^\ell_{ih}\,\delta^r_h$	

I call these measures ex-post because they assume that the necessary variations are observable and do not depend on underlying model primitives. First, I present the price variation in response to exogenous shocks and show that these effects propagate downstream through the cost of intermediate and final goods. Second, I characterize the first-order variation for the decentralized labor wedges and the labor income distribution. Third, I decompose the first-order variation for country-level TFP and establish a connection with the decentralized labor wedges that allow me to decompose the country-specific and distributional effects from the endogenous reallocation of labor across firms into variations of (i) exogenous distortions, (ii) endogenous variations in the expenditure distribution keeping the demand structure fixed, and (iii) endogenous recomposition in the demand structure from firms and households in response to relative price variations while keeping the expenditure distribution fixed.

4.1 Price Variation

Proposition 2 captures the network-adjusted response of prices to supply shocks. These shocks propagate downstream through the costs of intermediate inputs and final goods, and the cost-based firm-to-firm and firm-to-consumer centrality measures capture their magnitude.

Proposition 2. The change in sector i's prices, household h's price index, and country r's GDP deflator in response to productivity, markdown, and factor cost shocks are, to a first-order,

$$\begin{aligned}
d\log p_i &= -\sum_{j\in\mathscr{N}} \widetilde{\psi}^x_{ij}\, d\log\mathscr{A}_j\,\mu_j + \sum_{h\in\mathscr{H}} \widetilde{\psi}^\ell_{ih}\, d\log w_h, \\
d\log p^c_h &= -\sum_{i\in\mathscr{N}} \widetilde{\mathscr{B}}_{hi}\, d\log\mathscr{A}_i\,\mu_i + \sum_{b\in\mathscr{H}} \widetilde{\mathscr{C}}_{hb}\, d\log w_b, \\
d\log p_{Y_r} &= -\sum_{i\in\mathscr{N}} \ddot{\lambda}^r_i\, d\log\mathscr{A}_i\,\mu_i + \sum_{h\in\mathscr{H}} \ddot{\Lambda}^r_h\, d\log w_h,
\end{aligned}$$

where $d\log\mathscr{A}_i = d\log A_i + \omega^\ell_i\, d\log A^\ell_i + \omega^x_i\, d\log A^x_i + \sum_{h\in\mathscr{H}} \widetilde{\Omega}^\ell_{ih}\, d\log A^\ell_{ih} + \sum_{j\in\mathscr{N}} \widetilde{\Omega}^x_{ij}\, d\log A^x_{ij}$.

First, firm i's compound measure of productivity $d\log\mathscr{A}_i$ incorporates Hicks-neutral, labor-specific, and input-specific augmenting productivity shocks, and its effect on prices across all firms and households is isomorphic to an increase in the markdown for firm i. Second, labor costs have a direct effect on the labor bundle price that propagates through the supply of intermediate inputs to other firms and finally reaches consumption bundle prices. Third, the GDP deflator for country r depends negatively on productivity and markdown shocks and positively on wages. The elasticities from these shocks on the GDP deflator are equal to the country-specific shares of value added from a sector or worker.

4.2 Labor Wedges and the Income Distribution

Theorem 1 portrays the equilibrium characterization of the households' labor supply and the endogenous variation of the labor income distribution. This theorem represents an extension of the decentralized labor wedge decompositions from Rojas-Bernal (2023) to an open economy environment. For workers of type h, the labor wedge Γ_h gauges how the whole set of economic distortions influences their labor supply decision.

Theorem 1. In equilibrium, the labor supply from households of type h satisfies

$$\frac{U_{L_h}}{U_{C_h}} + \Gamma_h \frac{C_h}{L_h} = 0 \qquad \text{with} \qquad \Gamma_h = \frac{\Lambda_h}{\chi_h}. \tag{12}$$

The change of Λ_h in response to variations in the consumption distribution and consumer-to-worker centralities is, to a first-order,

$$d\,\Lambda_h = \overbrace{\sum_{b\in\mathscr{H}} \mathscr{C}_{bh}\, d\,\chi_b}^{\substack{\textit{Distributive}\\ \textit{Income}_h}} + \overbrace{\sum_{b\in\mathscr{H}} \chi_b\, d\,\mathscr{C}_{bh}}^{\substack{\textit{Income}\\ \textit{Centrality}_h}}, \tag{13}$$

$$\substack{\textit{Income}\\ \textit{Centrality}_h} = \overbrace{\sum_{i\in\mathscr{N}} \psi^{\ell}_{ih} \sum_{b\in\mathscr{H}} \chi_b\, d\,\beta_{bi}}^{\substack{\textit{Final Demand}\\ \textit{Recomposition}_h}} + \overbrace{\sum_{i\in\mathscr{N}} \psi^{\ell}_{ih} \sum_{j\in\mathscr{N}} \mu_j\, \lambda_j\, d\,\widetilde{\Omega}^{x}_{ji}}^{\substack{\textit{Intermediate Demand}\\ \textit{Recomposition}_h}} + \overbrace{\sum_{i\in\mathscr{N}} \mu_i\, \lambda_i\, d\,\widetilde{\Omega}^{\ell}_{ih}}^{\substack{\textit{Labor Demand}\\ \textit{Recomposition}_h}} + \overbrace{\sum_{i\in\mathscr{N}} \psi^{\ell}_{ih}\, \lambda_i\, d\, log\, \mu_i}^{\substack{\textit{Competitive}\\ \textit{Income}_h}}. \tag{14}$$

The decentralized labor wedge Γ_h from equation (12) relates the marginal rate of substitution between consumption and the labor supply with the household's average labor rate of transformation on consumption C_h/L_h. In equilibrium, the decentralized labor wedge equals the share of labor income to consumption expenditure, i.e., J_h/E_h. For an economy without distortions, labor compensation is the only source of income and $\Gamma_h = 1$.

Equation (13) divides the first-order variation of the labor income share into changes in the consumption distribution and changes in the consumer-to-worker centralities. First, *distributive income* captures how the revenue share for workers of type h increases as the expenditure share grows for households whose expenditure has a relatively high upstream centrality on their labor income. For example, Λ_h will increase in response to an endogenous redistribution of expenditure from type q to type b households if $\mathscr{C}_{bh} > \mathscr{C}_{qh}$. Second, *income centrality* portrays how the revenue share for workers of type h increases as the consumer-to-worker centralities on their labor income rise.

The income centrality variation collects four different effects. The *final* and *intermediate demand recomposition* characterize the effects of households' and firms' expenditure reallocation. These two channels convey that the labor revenue share for workers of type h will increase as the households' consumption patterns or the firms' cost structure shifts towards sectors with a high firm-to-worker centrality on their labor income. For example, Λ_h rises in response to a cost reallocation from sector j to sector i, by any firm or household, if $\psi^{\ell}_{ih} > \psi^{\ell}_{jh}$. The *labor demand recomposition* portrays the influence on the labor income share from higher labor demand; the magnitude of this effect is more prominent for big and relatively undistorted sectors. Finally, the *competitive income* tells us that lower profit margins in a sector will increase the labor income share for workers of type h in a magnitude proportional to the sector's size and the sector's centrality on the labor income of these workers.

4.3 Open Economy Decompositions

Theorem 2 characterizes aggregate country-level output Y in equilibrium and its first-order variation around the equilibrium.

Theorem 2. In equilibrium, country-level real GDP satisfies

$$Y_r = TFP_r \; F_r\left(\{L_h\}_{h\in\mathscr{H}}\right), \tag{15}$$

where TFP_r captures country r's total factor productivity and F_r satisfies $d\log F_r / d\log L_h = \ddot{\Lambda}^r_h$.

The change in Y_r and TFP_r are, to a first-order

$$d\log Y_r = d\log TFP_r + \sum_{h\in\mathscr{H}} \ddot{\Lambda}^r_h \, d\log L_h, \tag{16}$$

$$d\log TFP_r = \mathit{Technology}_r + \mathit{Competitiveness}_r - \mathit{Misallocation}_r, \tag{17}$$

where

$$\mathit{Technology}_r = \sum_{i\in\mathscr{N}} \ddot{\lambda}^r_i \; d\log \mathscr{A}_i, \qquad \mathit{Competitiveness}_r = \sum_{i\in\mathscr{N}} \ddot{\lambda}^r_i \; d\log \mu_i,$$

Misallocation has the following four equivalent definitions

$$1. \quad \overbrace{\sum_{h\in\mathscr{H}} \delta^r_h \; d\,\Lambda_h}^{\text{Entropic } TT_r} - d\log\Phi_r, \qquad 2. \quad \overbrace{\sum_{h\in\mathscr{H}} M^r_h \; d\,\chi_h}^{\text{Distributive } TT_r} + \overbrace{\sum_{h\in\mathscr{H}} \chi_h \sum_{b\in\mathscr{H}} \delta^r_b \; d\,\mathscr{C}_{hb}}^{\text{Centrality } TT_r} - d\log\Phi_r,$$

$$\begin{aligned} 3. \quad & \sum_{h\in\mathscr{H}} M^r_h \; d\,\chi_h + \overbrace{\sum_{h\in\mathscr{H}} \chi_h \sum_{i\in\mathscr{N}} F^r_i \; d\,\beta_{hi}}^{\text{Final Demand } TT_r} + \overbrace{\sum_{i\in\mathscr{N}} \mu_i\,\lambda_i \sum_{j\in\mathscr{N}} F^r_j \; d\,\widetilde{\Omega}^x_{ij}}^{\text{Intermediate Demand } TT_r} \\ & + \underbrace{\sum_{i\in\mathscr{N}} \mu_i\,\lambda_i \sum_{h\in\mathscr{H}} \delta^r_h \; d\,\widetilde{\Omega}^\ell_{ih}}_{} \!\!{}^{\text{Labor Demand } TT_r} + \overbrace{\sum_{i\in\mathscr{N}} \lambda_i \; F^r_i \; d\log\mu_i}^{\text{Competitive } TT_r} - d\log\Phi_r, \end{aligned}$$

and the variation for the GDP share is given by

$$\begin{aligned} d\,\Phi_r = & \sum_{h\in\mathscr{H}} \beta_{h|r} \; d\,\chi_h + \sum_{j\notin\mathscr{N}_r} \Omega^x_{j|r} \; d\,\lambda_j - \sum_{i\in\mathscr{N}_r} \left(\sum_{j\notin\mathscr{N}_r} \Omega^x_{ij} \right) d\,\lambda_i \\ & + \sum_{h\in\mathscr{H}} \chi_h \; d\,\beta_{h|r} + \sum_{j\notin\mathscr{N}_r} \lambda_j \; d\,\Omega^x_{j|r} - \sum_{i\in\mathscr{N}_r} \lambda_i \sum_{j\notin\mathscr{N}_r} d\,\Omega^x_{ij} \, . \end{aligned} \tag{18}$$

From equation (15), country r's real GDP in equilibrium is the product of TFP_r and a CRS function F_r that aggregates labor with elasticities equal to the country-level value-added weights $\ddot{\Lambda}_h^r$.

Equation (16) segments the output response into a TFP and a factoral component. Equation (17) divides the first-order variation of TFP into three components. First, *technology*$_r$ captures the direct effect of changes in productivity under a fixed allocation of resources. Second, *competitiveness*$_r$ portrays the reallocation effects from distortions in the absence of distributional variations on GDP shares and the labor income distribution. These two components tell us that in the absence of distributional reallocation, the effects on TFP of productivity and markdown changes in sector i are proportional to $\ddot{\lambda}_i^r$. Third, *misallocation*$_r$ represents the endogenous distributional losses in response to global GDP participation and income distribution changes.

Theorem 2 also contains three equivalent definitions for the *misallocation*$_r$ component, and each one gives us a different intuition about the effects on TFP_r from distributional changes. All three definitions capture the idea that the global allocation is more favorable to country r as their share of global GDP increases or when the new allocation of workers and intermediate inputs is more favorable for their production.

In the first definition, the *entropic terms of trade*$_r$ capture a reduction in the statistical distance between country r's value added distribution $\ddot{\Lambda}^r = \left\{\ddot{\Lambda}_h^r\right\}_{h\in\mathscr{H}}$ and the global labor income distribution $\Lambda = \{\Lambda_h\}_{h\in\mathscr{H}}$.[3] From the perspective of country r, worker b is relatively overvalued compared to worker h when $\delta_h^r > \delta_b^r$. The new allocation is more favorable for country r as labor income shifts from type h to type b workers and $\delta_h^r > \delta_b^r$. This effect portrays how more essential workers for domestic production are becoming relatively more affordable, allowing them to reallocate in response to higher labor demand from firms in supply chains relevant to the country r.

The last two definitions require the labor income share variations from Theorem 1. The second definition splits *misallocation*$_r$ into variations in the consumption distribution and consumer-to-worker centralities. First, the *distributive terms of trade*$_r$ imply that labor misallocation worsens as expenditure shifts towards households with high country r *expenditure efficiency*. Consumers of type h have a high country r *expenditure efficiency* M_h^r when the dot product of their vector of consumer-to-worker centralities $\mathscr{C}_{\uparrow h} = (\mathscr{C}_{h1}, \ldots, \mathscr{C}_{hH})'$ and the vector of country r's distortion centralities $\delta^r = (\delta_1^r, \ldots, \delta_H^r)'$ is high. High consumer-to-worker centralities imply that the consumption bundle from a household relies heavily on goods produced by relatively undistorted

[3] *Entropic* $TT_r = -d\mathscr{K}\left(\ddot{\Lambda}^r, \Lambda\right)$ where $d\mathscr{K}\left(\ddot{\Lambda}^r, \Lambda\right)$ stands for the first-order variation in response to changes in the distribution Λ for the Kullback-Leibler divergence or relative entropy measure $\mathscr{K}\left(\ddot{\Lambda}^r, \Lambda\right) = \sum_{h\in\mathscr{H}} \ddot{\Lambda}_h^r \, log\left(\Lambda_h/\ddot{\Lambda}_h^r\right)$. A more detailed explanation for how $\mathscr{K}(a,b)$ is a measure for the statistical distance between the distributions a and b can be found in Rojas-Bernal (2023).

supply chains. Hence, a high M_h^r implies that households of type h demand goods produced by firms within efficient supply chains that rely heavily on workers essential for the country r's domestic production. *Misallocation*$_r$ increases with χ_h when M_h^r is high because aggregate expenditure flows towards efficient firms that demand labor from workers that are essential for country r, reallocating workers away from sectors that are important for the domestic production of country r. The vector of country r's *expenditure efficiencies* $M^r = (M_1^r, \ldots, M_H^r)'$ is a sufficient statistic for the effect of expenditure distributional variations on TFP_r. For example, TFP_r will improve in response to an endogenous redistribution of expenditure from type h to type b households if $M_h^r > M_b^r$. Second, the *centrality terms of trade*$_r$ indicate that misallocation worsens as the consumer-to-worker centralities from a household increase, and the magnitude of this effect is more prominent when it takes place on workers with high country r distortion centralities. Workers of type h have a high country r distortion centrality δ_h^r when the labor income they receive is low compared to the value-added they provide for country r.

The last definition separates the *centrality terms of trade*$_r$ into four different effects that capture endogenous demand recomposition. The *final demand terms of trade*$_r$ and *intermediate demand terms of trade*$_r$ represent how misallocation worsens with an increase in the demand for goods produced by firms with high country r *revenue efficiency*. Firms in sector i have a high country r *revenue efficiency* F_i^r when the dot product of their firm-to-worker centralities $\left(\psi_{i1}^\ell, \ldots, \psi_{1H}^\ell\right)'$ and the vector of country r distortion centralities δ^r is high. High firm-to-worker centralities imply that the firm faces high markdowns or the intermediate input bundle relies heavily on goods produced by relatively undistorted supply chains. Hence, a high F_i^r implies that firms of type i produce within relatively efficient supply chains and require, directly or indirectly, on workers essential for the country r's domestic production. The *labor demand terms of trade*$_r$ portray how misallocation increases as the demand for high country r distortion centrality workers from big and relatively undistorted sectors rises. Finally, the *competitive terms of trade*$_r$ capture the effects on TFP_r from the reallocation of workers in response to variations in labor demand driven by markdowns. The vector of country r's *revenue efficiencies* $F^r = (F_1^r, \ldots, F_N^r)'$ is a sufficient statistic for the effect of final demand, intermediate demand, and markdown variations on TFP_r. For example, assume a markdown reduction in sector i of 1% such that the country-level GDP distribution, expenditure distribution, and the final, intermediate, and labor demand terms of trade are inelastic. In response to this shock, distributional *misallocation*$_r$ will fall by $\lambda_i\, F_i^r$, and total TFP_r will increase by $\lambda_i\, F_i^r - \ddot{\lambda}_i^r$. Hence, under these assumptions, a reduction in the markdown from sector i improves the country-level efficiency wedge when $F_i^r > \ddot{\lambda}_i^r/\lambda_i$.

Equation (18) characterizes the variation for the country r's global GDP share. First, the GDP share for country r increases as expenditure shifts towards households or foreign firms with a high expenditure intensity on domestic goods. For example, Φ_r increases as expenditure shifts from type b to type h consumers if $\beta_{h|r} > \beta_{b|r}$, or as sales shift from foreign sector i to foreign sector j if $\Omega^x_{i|r} > \Omega^x_{j|r}$. Second, the GDP share for country r falls as the sales share for domestic firms rises, and this effect is proportional to the intermediate input expenditure on foreign inputs. For example, Φ_r falls by $\sum_{j\in\mathscr{N}_r}\Omega^x_{ij}$ as the global sales share for the domestic sector i rises. Third, Φ_r increases with the share of expenditure on domestic goods from households and foreign firms. Finally, the Φ_r falls with the domestic firm intensity on foreign inputs.

Corollary 1. $d\,log\,TFP_r$ around the efficient equilibrium. In the absence of distortions

$$d\,log\,TFP_r = \overbrace{\sum_{i\in\mathscr{N}_r}\frac{\lambda_i}{\Phi_r}d\,log\,\mathscr{A}_i}^{Technology_r} + \overbrace{\sum_{i\in\mathscr{N}_r}\frac{\lambda_i}{\Phi_r}d\,log\,\mu_i - \sum_{i\in\mathscr{N}}\frac{\lambda_i}{\Phi_r}\sum_{h\in\mathscr{H}_r}\kappa_{ih}\,d\,log\,\mu_i}^{Competitiveness_r}$$
$$+ \overbrace{\frac{1}{\Phi_r}\sum_{h\in\mathscr{H}_r}d\,\Lambda_h - \sum_{h\in\mathscr{H}}\delta^r_h\,d\,\Lambda_h}^{-Misallocation_r}.$$

Corollary 1 characterizes the local variation for country-level TFP around the undistorted global allocation, i.e., $\mu = \mathbb{1}_N$. First, the value-added a country captures depends exclusively on the value-added domestic firms produce. Hence $\Phi_r\,\ddot{\lambda}^r_i = \mathbb{1}\{i\in\mathscr{N}_r\}\,\lambda_i$. This result implies that only domestic firms' productivity shocks directly influence country-level TFP. Second, markdown shocks in domestic and foreign firms can directly affect the country-level efficiency wedge. $Competitiveness_r$ falls in response to a 1% markdown reduction for the domestic sector i, and its elasticity equals $-\frac{\lambda_i}{\Phi_r}\left(1-\sum_{h\in\mathscr{H}_r}\kappa_{ih}\right)$. On the one hand, lower input demand from domestic firms reduces TFP and allows firms to create a profit margin. On the other hand, a fraction $\sum_{h\in\mathscr{H}_r}\kappa_{ih}$ of the additional profits are distributed to domestic households, increasing GNI_r. $Competitiveness_r$ increases in response to a 1% markdown reduction for the foreign sector i, and its elasticity equals $\frac{\lambda_i}{\Phi_r}\sum_{h\in\mathscr{H}_r}\kappa_{ih}$. This positive effect captures the GNI_r increase from additional profits distributed to domestic households. Finally, an increase in the labor income share of one unit for a domestic worker has an effect on $Misallocation_r$ equal to $\delta^r_h - \frac{1}{\Phi_r}$. This effect is positive as long as $\ddot{\Lambda}^r_h > \frac{\Lambda_h}{\Phi_r}$, which implies that the new allocation makes country r worse-off as long as the value-added captured by country r is large enough. Consequently, TFP_r can improve in response to an increase in the labor income share for a domestic household as long as their production has a small value-added effect on domestic production. In other words, making more expensive domestic factors is good for a country if those factors produce value-added for foreign economies. An increase in the labor income share of one unit for a foreign worker increases misallocation by δ^r_h.

Corollary 2. Open economy Hulten's theorem. In the absence of distortions, with country-specific labor markets, and with full equity home bias

$$d\,log\,TFP_r = \sum_{i\in\mathscr{N}_r}\frac{\lambda_i}{\Phi_r}d\,log\,\mathscr{A}_i.$$

Corollary 2 characterizes the local variation for country-level TFP around the undistorted global allocation when factor markets are domestic and there is full equity home bias. This is a Hulten (1978) theorem type of result for an open economy that characterizes the country-specific envelope condition for the efficiency wedge. A symmetric domestic productivity shock of 1% has an effect on the country-level efficiency wedge equal to the network multiplier ξ_r

$$d\,log\,TFP_r = \xi_r = \frac{\sum_{i\in\mathscr{N}_r} \lambda_i}{\sum_{i\in\mathscr{N}_r} \omega_i^\ell \lambda_i} \geq 1.$$

The difference between Corollary 2 and Hulten's (1978) is that in the latter, the system of equations for the Domar weights depends exclusively on domestic demand because the economy is closed, i.e., $\lambda_i = \sum_{h\in\mathscr{H}_r} \mathscr{B}_{hi}\,\chi_h + \sum_{j\in\mathscr{N}_r} \Omega^x_{ji}\,\lambda_j \;\forall i \in \mathscr{N}_r$, $\chi_h = \sum_{i\in\mathscr{N}_r} \left(\Omega^\ell_{ih} + \Omega^\pi_{ih}\right)\lambda_i \;\forall h \in \mathscr{H}_r$, and $\xi_r = \sum_{i\in\mathscr{N}_r} \lambda_i$.

Theorem 3 reports the decomposition for global TFP and its relationship with country-level TFP.

Theorem 3. In equilibrium, real GDP satisfies

$$Y = TFP\;F\left(\{L_h\}_{h\in\mathscr{H}}\right), \tag{19}$$

where TFP captures global total factor productivity and F satisfies $d\,log\,F/d\,log\,L_h = \widetilde{\Lambda}_h$.

The change in Y and TFP are, to a first-order

$$d\,log\,Y = d\,log\,TFP + \sum_{h\in\mathscr{H}} \widetilde{\Lambda}_h\, d\,log\,L_h,$$

$$d\,log\,TFP = \mathit{Technology} + \mathit{Competitiveness} - \mathit{Misallocation},$$

where

$$\mathit{Technology} = \sum_{i\in\mathscr{N}} \widetilde{\lambda}_i\; d\,log\,\mathcal{A}_i, \qquad \mathit{Competitiveness} = \sum_{i\in\mathscr{N}} \widetilde{\lambda}_i\; d\,log\,\mu_i,$$

and *Misallocation* has the following three equivalent definitions

$$1.\quad \overbrace{\sum_{h\in\mathscr{H}} \delta_h\; d\,\Lambda_h}^{\text{Entropic TT}}, \qquad 2.\quad \overbrace{\sum_{h\in\mathscr{H}} M_h\; d\,\chi_h}^{\text{Distributive TT}} + \overbrace{\sum_{h\in\mathscr{H}} \chi_h \sum_{b\in\mathscr{H}} \delta_b\; d\,\mathscr{C}_{hb}}^{\text{Centrality TT}},$$

$$3.\ \sum_{h\in\mathscr{H}} M_h\, d\chi_h + \overbrace{\sum_{h\in\mathscr{H}}\chi_h \sum_{i\in\mathscr{N}} F_i\, d\beta_{hi}}^{\textit{Final Demand TT}} + \overbrace{\sum_{i\in\mathscr{N}}\mu_i\,\lambda_i \sum_{j\in\mathscr{N}} F_j\, d\widetilde{\Omega}^x_{ij}}^{\textit{Intermediate Demand TT}}$$
$$+ \overbrace{\sum_{i\in\mathscr{N}}\mu_i\,\lambda_i \sum_{h\in\mathscr{H}} \delta_h\, d\widetilde{\Omega}^\ell_{ih}}^{\textit{Labor Demand TT}} + \overbrace{\sum_{i\in\mathscr{N}}\lambda_i\, F_i\, d\log\mu_i}^{\textit{Competitive TT}}.$$

The relationship between TFP and TFP_r is given by

$$d\log TFP = \sum_{r\in\mathscr{R}} \Phi_r\, d\log TFP_r. \tag{20}$$

These results coincide with the decomposition for real GDP in Rojas-Bernal (2023), and the interpretation from these equations is in that paper. Equation (20) is novel and shows that country-level TFP is a segmentation of global TFP. Hence, country level $misallocation_r$ captures a decomposition of global distributional $misallocation$. Reductions in $misallocation_r$ represent distributional gains that are favorable to country r, and these gains are not necessarily driven by higher variations in global $misallocation$. For example, from Hulten (1978), we know that around the efficient equilibrium, the effect from productivity shocks on global TFP is given by $d\log TFP = \lambda'\, d\log \mathcal{A}$ and $misallocation = 0$. However, there is still space for redistributive effects on country-level TFP, i.e., $misallocation_r \neq 0$.

5 Difference with Baqaee & Farhi (2023)

Baqaee & Farhi (2023) introduce a country-level TFP decomposition for a general production network open economy with distortions, where domestic firms exclusively use domestic factors from a country, and where profits generated by domestic firms are transfered exclusively to domestic households. The notation and proofs from Baqaee & Farhi (2023) rely on different statistics and assumptions. Section 3 in the Online Appendix proves the equivalence between the decomposition from Baqaee & Farhi (2023) and Theorem 2 once the country-specific factoral market segementation constraint is imposed. In other words, the decompositions from Baqaee & Farhi (2023) are a specific case from the results introduced in Section 4. In this section, I present the decomposition from Baqaee & Farhi (2023) and establish how their decomposition is a limiting case from Theorem 2.

Let me start by defining the net quantity of good $i \in \mathscr{N}$ produced by country $r \in \mathscr{R}$

$$q_{ri} = y_i \mathbb{1}\{i\in\mathscr{N}_r\} - \sum_{j\in\mathscr{N}_r} x_{ji} = \begin{cases} y_i - \sum_{j\in\mathscr{N}_r} x_{ji} & \text{if} \quad i\in\mathscr{N}_r \\ -\sum_{j\in\mathscr{N}_r} x_{ji} & \text{if} \quad i\notin\mathscr{N}_r \end{cases}.$$

From here, the share of good q_{ri} in the final output of country r is given by

$$\Omega_{Y_r i} = \frac{p_i q_{ri}}{GDP_r}.$$

This allows them to characterize the changes for the GDP deflator and real GDP for country r as

$$\widehat{p}_{Y_r} = \sum_{i \in \mathscr{N}} \Omega_{Y_r i}\, \widehat{p}_i \qquad \text{and} \qquad \widehat{Y}_r = \sum_{i \in \mathscr{N}} \Omega_{Y_r i}\, \widehat{q}_{ri}.$$

The first-order variation for domestic prices is given by

$$\widehat{p}_{i \in \mathscr{N}_r} = \left(I_{N_r} - \widetilde{\Omega}_x^{D_r} \right)^{-1} \left(\widetilde{\Omega}_\ell^r \widehat{w} + \widetilde{\Omega}_x^{M_r} \widehat{p}_{i \notin \mathscr{N}_r} - \widehat{\mathscr{A}} - \widehat{\mu} \right),$$

where $\widetilde{\Omega}_x^{D_r}$ is the $N_r \times N_r$ domestic cost-based input-output matrix, $\widetilde{\Omega}_x^{M_r}$ is the $N_r \times (N - N_r)$ imported cost-based input-output matrix, $\widetilde{\Omega}_\ell^r$ is the $N_r \times H$ domestic cost-based factor matrix, $\widehat{p}_{i \in \mathscr{N}_r}$ is a vector of dimension N_r that captures the variation for domestic prices, and $\widehat{p}_{i \notin \mathscr{N}_r}$ is a vector of dimension $N - N_r$ that captures the variation for foreign prices. Notice that $\widetilde{\Omega}_x^{D_r}$ and $\widetilde{\Omega}_x^{M_r}$ are coming from a reorganization of the rows in $\widetilde{\Omega}_x$ that characterize the intermediate input demand for firms that operate in country r, and $\widetilde{\Omega}_\ell^r$ is composed of the rows in $\widetilde{\Omega}_\ell$ that characterize the primary factor demand for firms that operate in country r.

Additionally let me introduce the following definitions used by Baqaee & Farhi (2023).

1. $\widetilde{\psi}_{ij}^{x_r}$ represents the ij element of matrix $\left(I_{N_r} - \widetilde{\Omega}_x^{D_r} \right)^{-1}$.
2. For sector j, $\widetilde{\lambda}_{Y_r j} = \sum_{i \in \mathscr{N}_r} \Omega_{Y_r i} \widetilde{\psi}_{ij}^{x_r}$.
3. For factor $h \in \mathscr{H}$, $\widetilde{\Lambda}_{Y_r h} = \sum_{i \in \mathscr{N}_r} \Omega_{Y_r i} \sum_{j \in \mathscr{N}_r} \widetilde{\psi}_{ij}^{x_r} \widetilde{\Omega}_{jh}^{\ell}$.
4. For foreign sector $i \notin \mathscr{N}_r$, $\widetilde{\Lambda}_{Y_r i} = \sum_{m \in \mathscr{N}_r} \Omega_{Y_r m} \sum_{j \in \mathscr{N}_r} \widetilde{\psi}_{mj}^{x_r} \widetilde{\Omega}_{ji}^{x}$.

Theorem 4. Baqaee & Farhi (2023). Under the segmentation of factoral markets and rebated income by country, the change in TFP_r is, to a first-order

$$\widehat{TFP}_r = \sum_{j \in \mathscr{N}_r} \widetilde{\lambda}_{Y_r j} \left(\widehat{\mathscr{A}}_j + \widehat{\mu}_j \right) - \sum_{h \in \mathscr{H}_r} \widetilde{\Lambda}_{Y_r h}\, \widehat{\Lambda}_{Y_r h} + \sum_{i \notin \mathscr{N}_r} \left(\widetilde{\Lambda}_{Y_r i} - \Lambda_{Y_r i} \right) \left(\widehat{q}_{ri} - \widehat{\Lambda}_{Y_r i} \right), \tag{21}$$

with $\Lambda_{Y_r i} = -\frac{p_i q_{ri}}{GDP_r}$ for $i \notin \mathscr{N}_r$.

Theorem 4 is a particular case of Theorem 2 with no reallocation of labor across countries and with full equity home bias. However, the differences go beyond the country-level segmentation of factoral and equity markets. First, equation 17 characterizes the effect of productivity and markdown shocks from all firms. In contrast, equation equation 21 only captures the effect from domestic firms. The effect from foreign firms takes place through the last component of the equation, which captures the real variation of the net quantity of imported goods. Second, for equation equation 17 it is not necessary to capture any variation for the real allocation of goods between countries, while in equation equation 21 it is necessary to observe the net quantity of goods imported and their variation. This last restriction is empirically relevant, as many input-output databases do not have a sectoral prices index that allows the identification of real quantities, e.g., the Bureau of Economics Analysis input-output network.

6 Quantitative Illustration

In this section, I study one particular quantitative application of my decompositions: the open-economy Hulten's theorem as an approximation for country-level TFP. Without distortions, the model from Section 2 requires measures for three types of money flows: (1) firm-to-firm in the supply of intermediate inputs, (2) firm-to-workers in the supply of labor, and (3) consumer-to-firm in the supply of final goods. I calibrate the model to the long-run world input-output database (Woltjer et al., 2021) and the Penn World tables (Feenstra et al., 2015). I examine the model's country-level efficiency wedge implications. The objective is to evaluate if the open economy Hulten's theorem from Corollary 2 is a good measure for country-level TFP dynamics.

6.1 Data and Calibration

The long-run world input-output database covers the period 1965 to 2000. It provides a detailed input-output matrix for 23 sectors in 25 countries and the rest of the world. On the production side, it captures two dimensions of heterogeneity: (i) sectoral heterogeneity in the demand for intermediate inputs across all sectors in the global economy and (ii) sectoral heterogeneity in the demand for primary factors. Additionally, for each country, there are measures of the final expenditure intensity across sectors. Hence, under the assumptions of a country-level representative household, a single country-specific factor (labor), and complete equity home bias, household heterogeneity has three dimensions: (i) heterogeneity in the sources of factoral income, (ii) heterogeneity in the sources of rebated profits, and (iii) heterogeneity in their consumption expenditure intensity.

One feature of the long-run world input-output database is that there is no decomposition of the value-added extracted by a sector. Hence, by imposing the assumption of no distortions, extracted value added corresponds to factoral income, and there are no profits on equilibrium. These assumptions allow me to calibrate the for all years t from 1965 to 2000 the following parameters for all households $h \in \mathscr{R}$ and for all sectors $i \in \mathscr{N}_r$.

$$\omega^{\ell}_{i,t} = \frac{\text{Value Added}_{i,t}}{\text{Total Costs}_{i,t}}, \quad \omega^{x}_{i,t} = \frac{\text{Intermediate Costs}_{i,t}}{\text{Total Costs}_{i,t}} \quad \alpha_{ir,t} = 1, \quad \Omega^{x}_{ij,t} = \frac{\text{Sales from } j \text{ to } i_t}{\text{Intermediate Cost}_{i,t}},$$

$$\text{Total Cost}_{i,t} = \text{Value Added}_{i,t} + \text{Intermediate Cost}_{i,t}, \qquad \text{Value Added}_i = \text{Labor Costs}_{i,t},$$

$$\text{Sales}_{i,t} = \text{Total Cost}_{i,t}, \qquad \beta_{ri,t} = \frac{\text{Sales from } j \text{ to } i_t}{\text{GDP}_{r,t}}, \qquad \text{GDP}_{r,t} = \sum_{i \in \mathscr{N}_r} \text{Value Added}_{i,t}.$$

The world input-output database also provides a price index $p_{i,t}$ for the goods from each sector. Using this index and nominal flows, one can estimate real quantities. I obtain a country-level yearly estimate of labor force participation $L_{r,t}$ from the Penn World tables. This measure allows me to estimate a country-level yearly wage $w_{r,t} = GDP_{r,t}/L_{r,t}$.

6.2 Sectoral Solow Residuals

The assumption of no distortions allows me to use the sectoral Solow (1957) residual decomposition for an input-output economy introduced by Caves et al. (1982) and Jorgenson et al. (1987). This decomposition has been more recently implemented by Fadinger et al. (2022) and McNerney et al. (2022). This decomposition assumes that the global economy is at an efficient equilibrium and markdown variations are null. Productivity shocks for sector $i \in \mathscr{N}$ are given by

$$d\,log\,\mathscr{A}_{i,t} = -\,\omega^{\ell}_{i,t-1}\,d\,log\,\frac{\ell_{ir,t}}{y_{i,t}} - \,\omega^{x}_{i,t-1}\sum_{j \in \mathscr{N}} \omega^{x}_{ij,t-1}\,d\,log\,\frac{x_{ij,t}}{y_{i,t}},$$

with $d\,log\,\frac{\ell_{ir,t}}{y_{i,t}} = d\,log\,\omega^{\ell}_{i,t} - d\,log\,\frac{w_{r,t}}{p_{i,t}}$ and $d\,log\,\frac{x_{ij,t}}{y_{i,t}} = d\,log\,\Omega^{x}_{ij,t} - d\,log\,\frac{p_{j,t}}{p_{i,t}}$.

Figures 1 shows the productivity levels for the 23 sectors in China and the United States. I normalize the 1965 levels of productivity at 100. There has been plenty of heterogeneity in sectoral productivity shocks for both countries. On the one hand, China's technology was mainly driven by productivity shocks in the manufacturing, and electrical and optical equipment sectors. On the other hand, the US's technology was primarily driven by shocks in the electrical and optical equipment and secondarily by productivity shocks in the telecommunication and retail sectors. I want to highlight the Moore's Law type of exponential growth for productivity shocks in the electrical and optical equipment sectors both for China and the US. Furthermore, consistent technological growth in China only started around the middle of the 80s, which coincides with the eve of their first wave of globalization.

6.3 Results

Corollary 2 tells us that

$$d\,log\,TFP_{r,t} = \sum_{i \in \mathcal{N}_r} \frac{\lambda_{i,t-1}}{\Phi_{r,t-1}}\, d\,log\,\mathcal{A}_{i,t}.$$

I will compare these estimates with a rough measure of country-level TFP estimated using the the Penn World Tables difference between the growth in real GDP and the labor force participation, i.e.,

$$d\,log\,TFP^*_{r,t} = d\,log\,\frac{Y_{r,t}}{L_{r,t}}.$$

Table 4 shows the Pearson correlation coefficients between $d\,log\,TFP_r$ and $d\,log\,TFP^*_r$. The average correlation coefficient is 0.69. The lowest is Taiwan with 0.21, and the highest is the US with 0.92. Figures 2, 3, 4, 5, and 6 show $d\,log\,TFP_{r,t}$ and $d\,log\,TFP^*_{r,t}$ for the countries in the sample. Both correlations and graphs allow me to say that Corollary 2 captures a good empirical representation of the actual country-level TFP growth.

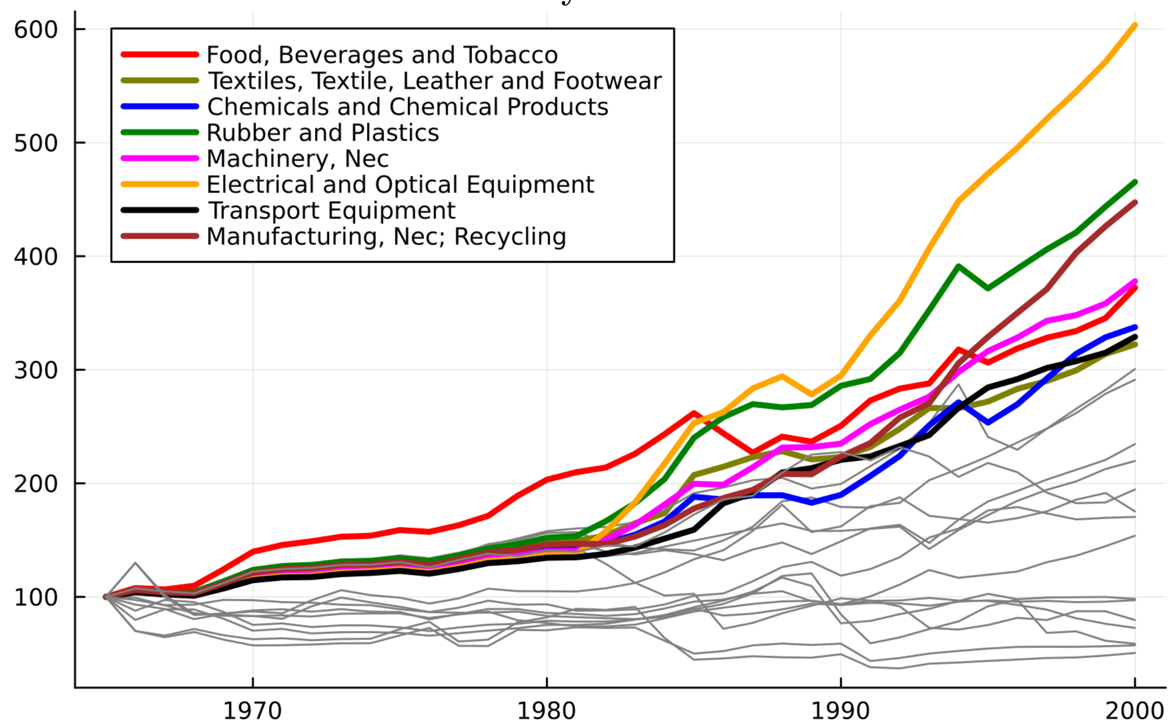
7 Conclusion

This paper develops a general aggregation theory for a production network open economy with distortions and endogenous labor supply. I provide decompositions for country-level TFP that explain how international intermediate input markets allow for cross-country spillovers in productivities, distortions, and labor income distribution variations. The decompositions allow me to identify the sufficient statistics necessary to measure the first-order variation for country-level TFP. Among those statistics, I construct new measures that capture, for a specific country, the value-added contribution that any firm or worker has. From these statistics, I can identify how countries can capitalize on foreign intermediate inputs to produce domestic goods that yield surplus profits. Through this mechanism, foreign production and factors foreign firms use can directly contribute value-added to a country. Without distortion, with country-specific factor markets, and with complete equity home bias, I identify an open economy Hulten (1978) theorem type of result that characterizes the first-order variation for a country's TFP. Using data from the long-run world input-output database, I show that the latter decomposition highly correlated with observable and independent measures of country-level TFP.

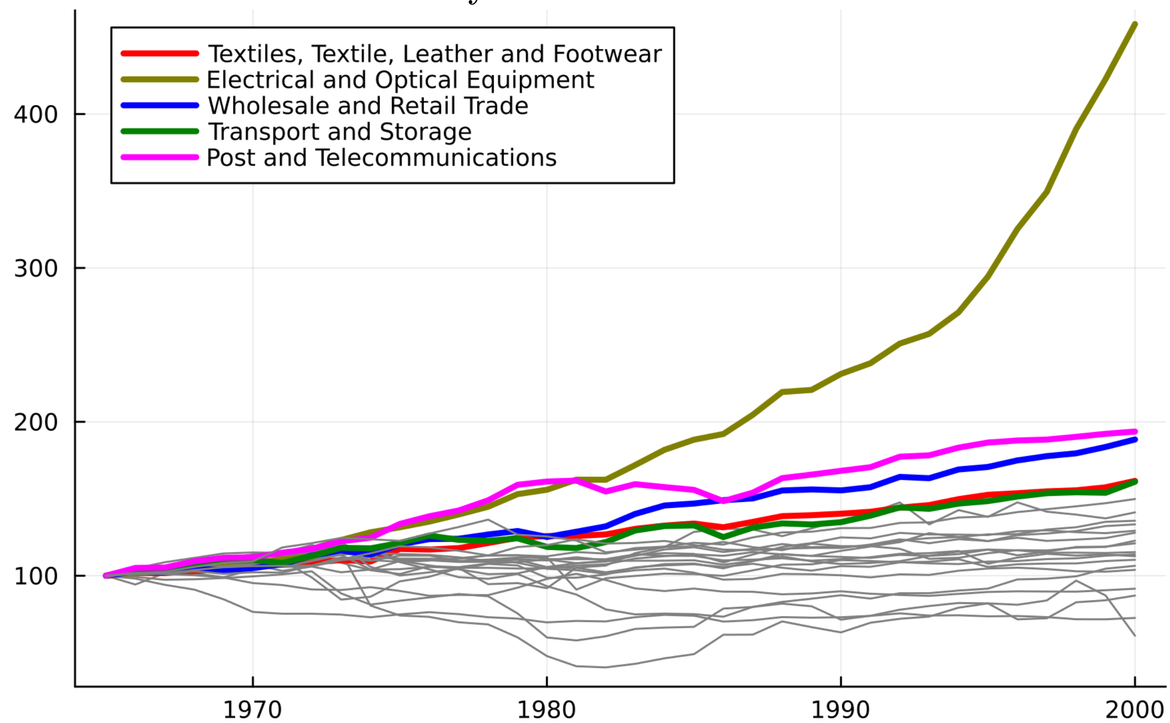
Figures and Tables

Figure 1: Sectoral Solow Residuals

A. Productivity Shocks in China



B. Productivity Shocks in the United States



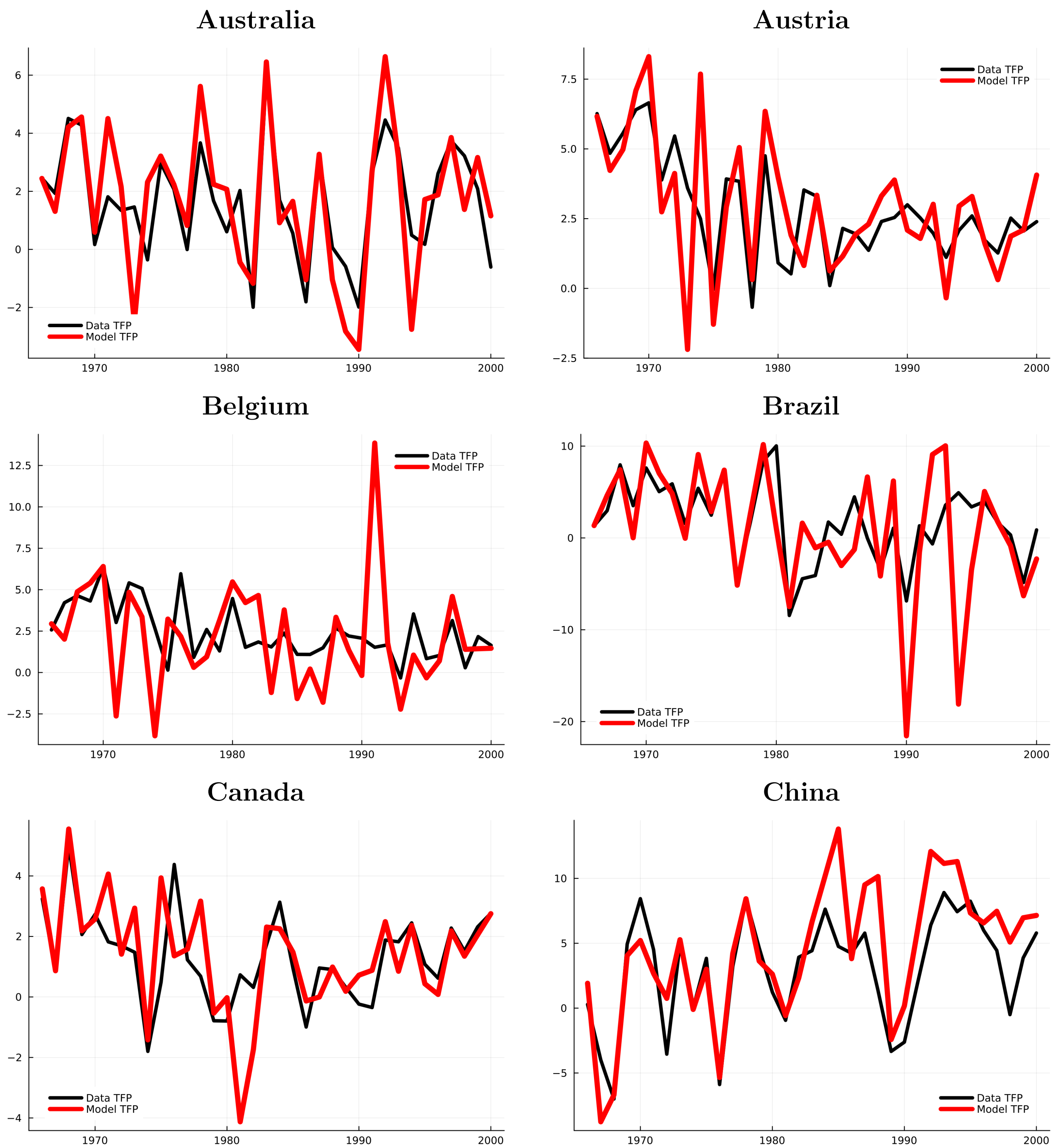
Note: Sectoral productivity levels for 1965 are normalized at 100.

Table 4: Pearson Correlation Coefficients between $dlogTFP_r$ and $dlogTFP_r^*$

Country	Correlation	Country	Correlation
Australia	0.79	India	0.73
Austria	0.67	Ireland	0.55
Belgium	0.35	Italy	0.71
Brazil	0.55	Japan	0.95
Canada	0.63	Korea	0.73
China	0.82	Mexico	0.50
Denmark	0.56	Netherlands	0.65
Finland	0.69	Portugal	0.81
France	0.82	Spain	0.81
Germany	0.79	Sweden	0.61
Great Britain	0.65	Taiwan	0.21
Greece	0.88	United States	0.93
Hong Kong	0.74		

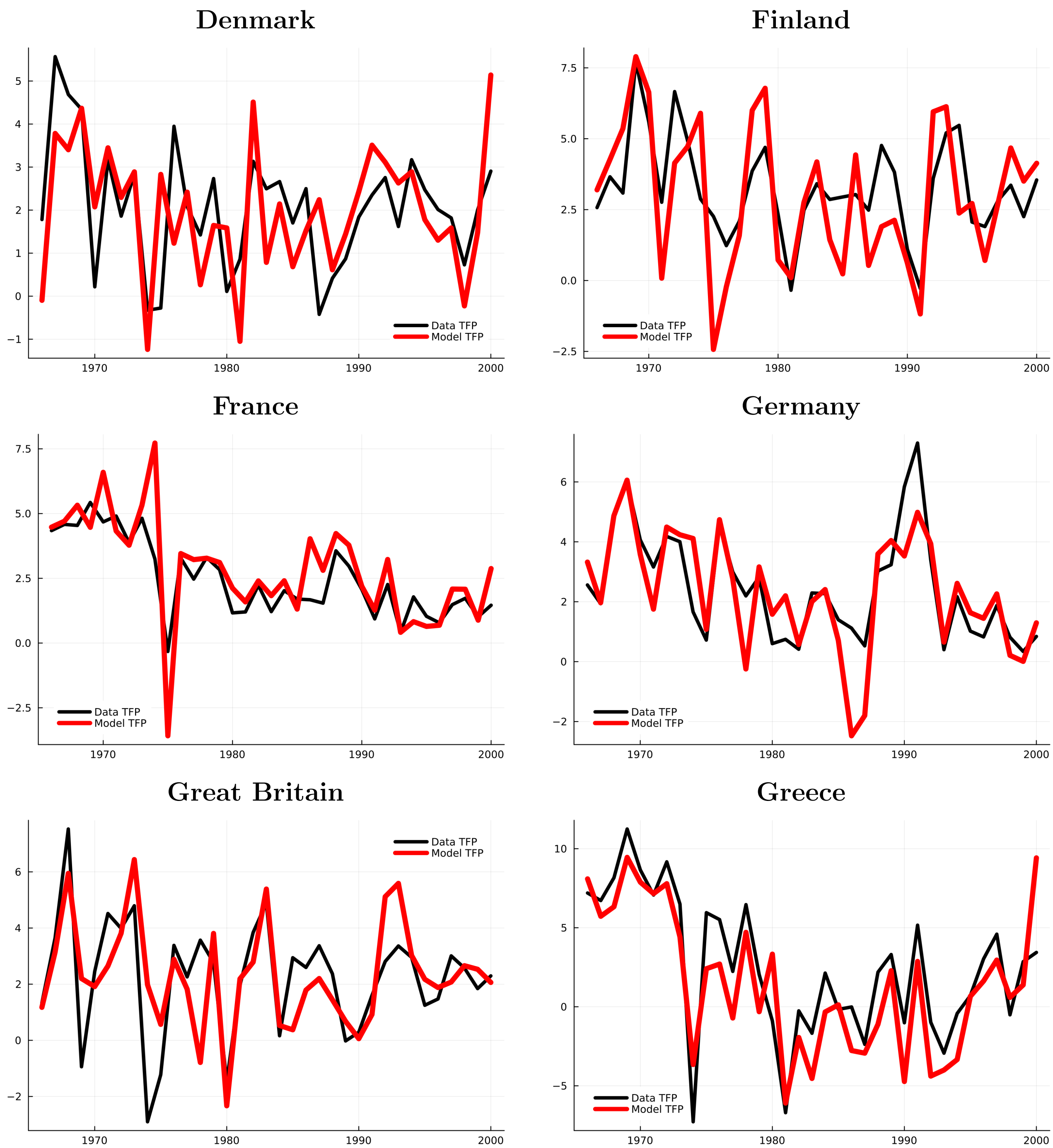
Note: Pearson correlation coefficient for each country between $dlogTFP_{r,t}$ and $dlogTFP_{r,t}^*$ between 1966 and 2000.

Figure 2: $dlogTFP_r$ and $dlogTFP^*_r$



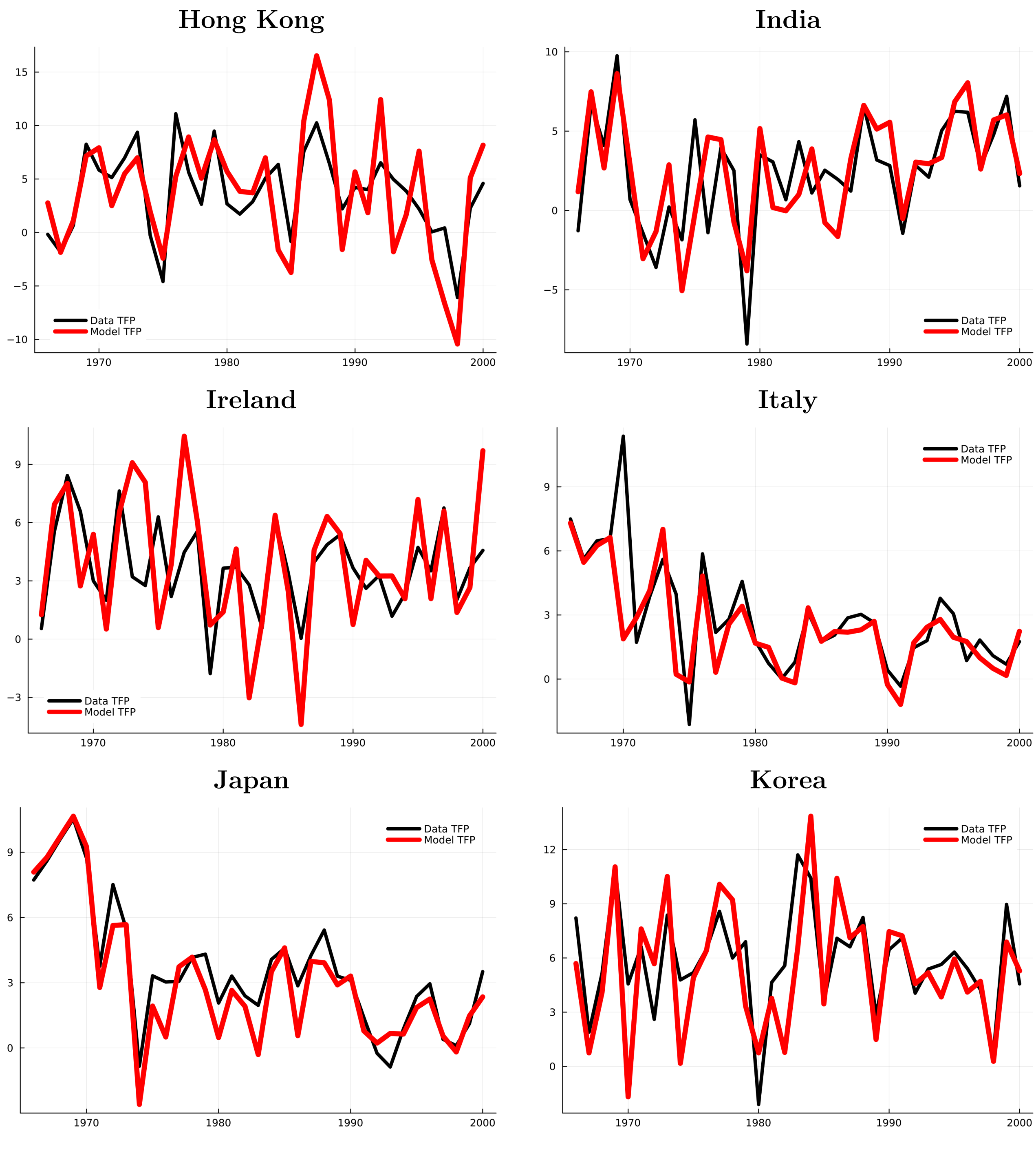
Note: Data TFP refers to $dlogTFP^* * 100$ and model TFP to $dlogTFP$ from Corollary 2

Figure 3: $dlogTFP_r$ and $dlogTFP_r^*$



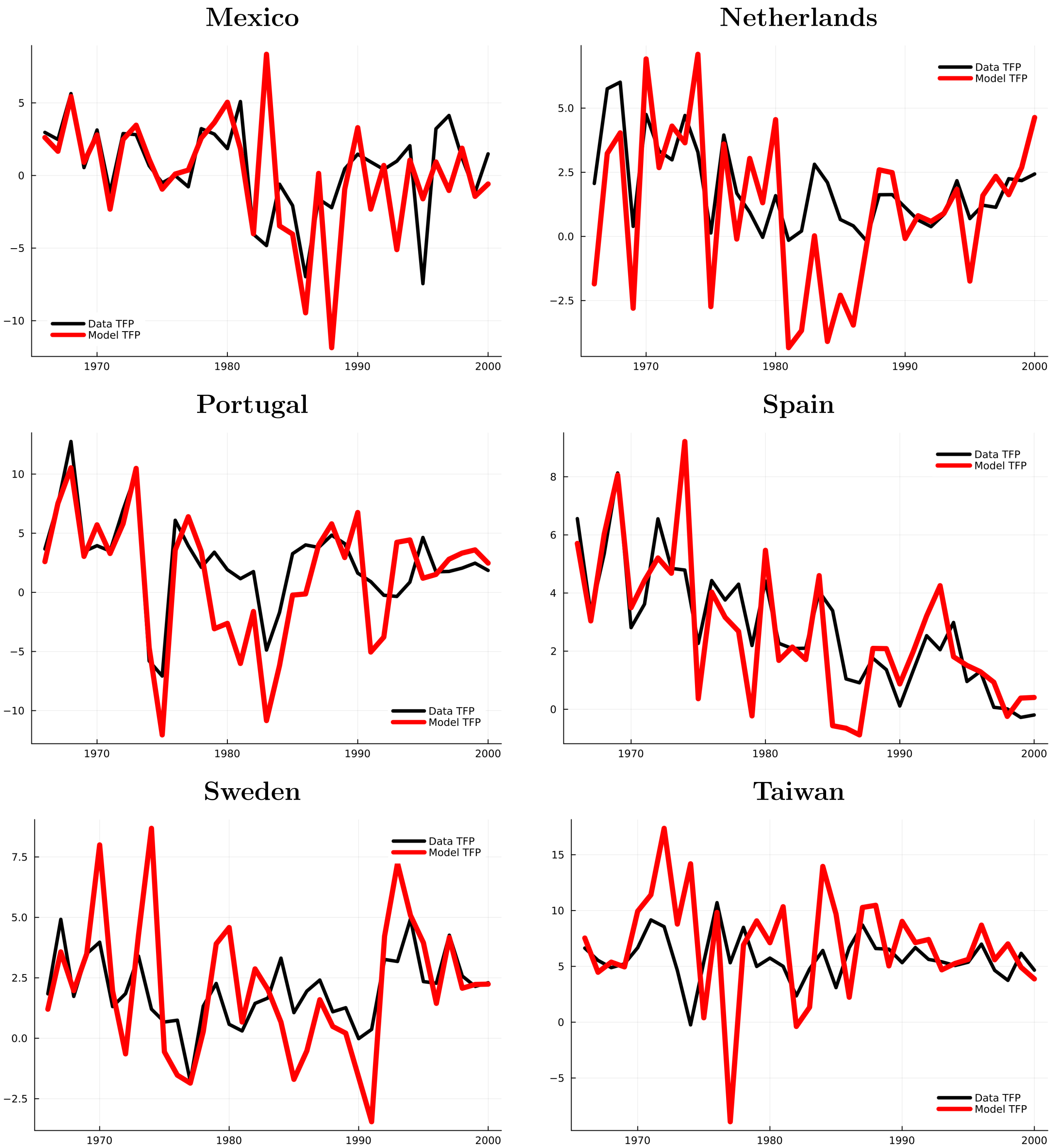
Note: Data TFP refers to $dlogTFP^* * 100$ and model TFP to $dlogTFP$ from Corollary 2

Figure 4: $\boldsymbol{dlogTFP_r}$ and $\boldsymbol{dlogTFP_r^*}$



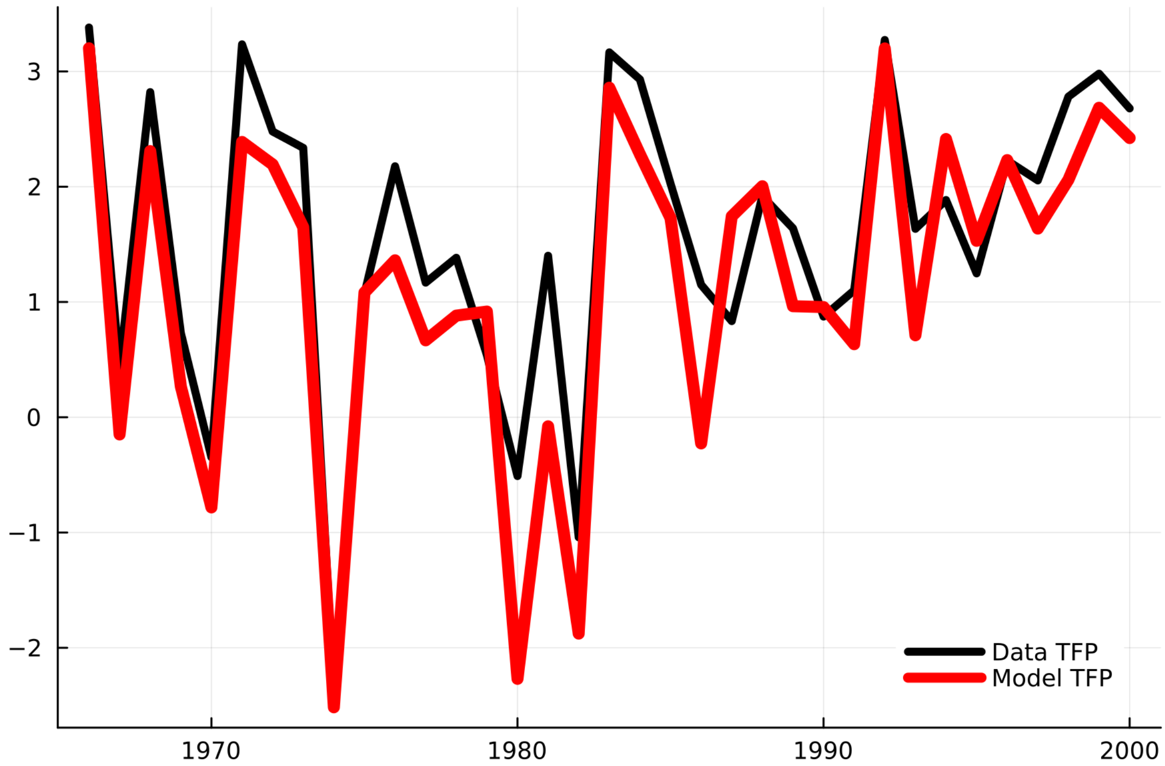
Note: Data TFP refers to $dlogTFP^* * 100$ and model TFP to $dlogTFP$ from Corollary 2

Figure 5: $dlogTFP_r$ and $dlogTFP_r^*$



Note: Data TFP refers to $dlogTFP^* * 100$ and model TFP to $dlogTFP$ from Corollary 2

Figure 6: $dlogTFP_r$ and $dlogTFP_r^*$ for the United States



Note: Data TFP refers to $dlogTFP^* * 100$ and model TFP to $dlogTFP$ from Corollary 2

References

Acemoglu, D., Akcigit, U., & Kerr, W. (2016). Networks and the macroeconomy: An empirical exploration. *NBER Macroeconomics Annual*, *30*(1), 273–335.

Acemoglu, D., Carvalho, V. M., Ozdaglar, A., & Tahbaz-Salehi, A. (2012). The network origins of aggregate fluctuations. *Econometrica*, *80*(5), 1977–2016.

Asker, J., Collard-Wexler, A., & De Loecker, J. (2014). Dynamic inputs and resource (mis) allocation. *Journal of Political Economy*, *122*(5), 1013–1063.

Baqaee, D. (2018). Cascading failures in production networks. *Econometrica*, *86*(5), 1819–1838.

Baqaee, D., & Farhi, E. (2020). Productivity and misallocation in general equilibrium. *The Quarterly Journal of Economics*, *135*(1), 105–163.

Baqaee, D., & Farhi, E. (2023). Networks, barriers, and trade.

Basu, S. (1995). Intermediate goods and business cycles: Implications for productivity and welfare. *The American Economic Review*, *85*(3), 512–531.

Basu, S., & Fernald, J. G. (2002). Aggregate productivity and aggregate technology. *European Economic Review*, *46*(6), 963–991.

Bigio, S., & La'O, J. (2020, 05). Distortions in Production Networks*. *The Quarterly Journal of Economics*, *135*(4), 2187-2253.

Carvalho, V. M., Nirei, M., Saito, Y. U., & Tahbaz-Salehi, A. (2021). Supply chain disruptions: Evidence from the great east japan earthquake. *The Quarterly Journal of Economics*, *136*(2), 1255–1321.

Caves, D. W., Christensen, L. R., & Diewert, W. E. (1982). The economic theory of index numbers and the measurement of input, output, and productivity. *Econometrica: Journal of the Econometric Society*, 1393–1414.

Ciccone, A. (2002). Input chains and industrialization. *The Review of Economic Studies*, *69*(3), 565–587.

Dixit, A. K., & Stiglitz, J. E. (1977). Monopolistic competition and optimum product diversity. *The American economic review*, *67*(3), 297–308.

Domar, E. D. (1961). On the measurement of technological change. *The Economic Journal*, *71*(284), 709–729.

Dupor, B. (1999). Aggregation and irrelevance in multi-sector models. *Journal of Monetary Economics*, *43*(2), 391–409.

Fadinger, H., Ghiglino, C., & Teteryatnikova, M. (2022). Income differences, productivity, and input-output networks. *American Economic Journal: Macroeconomics*, *14*(2), 367–415.

Feenstra, R. C., Inklaar, R., & Timmer, M. P. (2015). The next generation of the penn world table. *American economic review*, *105*(10), 3150–3182.

Foerster, A. T., Sarte, P.-D. G., & Watson, M. W. (2011). Sectoral versus aggregate shocks: A structural factor analysis of industrial production. *Journal of Political Economy*, *119*(1), 1–38.

Hall, R., & Diamond, P. (1990). Growth, productivity, unemployment: Essays to celebrate bob solow's birthday. *by P. Diamond. Cambridge: MIT Press. Chap. Invariance Properties of Solow's Productivity Residual*.

Horvath, M. (1998). Cyclicality and sectoral linkages: Aggregate fluctuations from independent sectoral shocks. *Review of Economic Dynamics*, *1*(4), 781–808.

Horvath, M. (2000). Sectoral shocks and aggregate fluctuations. *Journal of Monetary Economics*, *45*(1), 69–106.

Hsieh, C.-T., & Klenow, P. J. (2009). Misallocation and manufacturing tfp in china and india. *The Quarterly journal of economics*, *124*(4), 1403–1448.

Hulten, C. R. (1978). Growth accounting with intermediate inputs. *The Review of Economic Studies*, *45*(3), 511–518.

Huo, Z., Levchenko, A. A., & Pandalai-Nayar, N. (2021). *International comovement in the global production network* (Tech. Rep.). National Bureau of Economic Research.

Jones, C. I. (2011). Intermediate goods and weak links in the theory of economic development. *American Economic Journal: Macroeconomics*, *3*(2), 1–28.

Jones, C. I. (2013). Misallocation, economic growth, and input–output economics. In D. Acemoglu, M. Arellano, & E. Dekel (Eds.), *Advances in economics and econometrics: Tenth world congress* (Vol. 2, p. 419–456). Cambridge University Press. doi: 10.1017/CBO9781139060028.011

Jorgenson, D. W., Gollop, F. M., & Fraumeni, B. M. (1987). Productivity and us economic growth.

Liu, E. (2019). Industrial policies in production networks. *The Quarterly Journal of Economics*, *134*(4), 1883–1948.

Long, J. B., & Plosser, C. I. (1983). Real business cycles. *Journal of political Economy*, *91*(1), 39–69.

McKenzie, L. W. (1959). On the existence of general equilibrium for a competitive market. *Econometrica: journal of the Econometric Society*, 54–71.

McNerney, J., Savoie, C., Caravelli, F., Carvalho, V. M., & Farmer, J. D. (2022). How production networks amplify economic growth. *Proceedings of the National Academy of Sciences*, *119*(1), e2106031118.

Osotimehin, S. (2019). Aggregate productivity and the allocation of resources over the business cycle. *Review of Economic Dynamics*, *32*, 180–205.

Petrin, A., & Levinsohn, J. (2012). Measuring aggregate productivity growth using plant-level data. *The Rand journal of economics*, *43*(4), 705–725.

Restuccia, D., & Rogerson, R. (2008). Policy distortions and aggregate productivity with heterogeneous establishments. *Review of Economic dynamics*, *11*(4), 707–720.

Rojas-Bernal, A. (2023). Inequality and misallocation under production networks.

Solow, R. M. (1957). Technical change and the aggregate production function. *The review of Economics and Statistics*, 312–320.

Woltjer, P., Gouma, R., & Timmer, M. P. (2021). Long-run world input-output database: Version 1.0 sources and methods.

Yi, K.-M. (2003). Can vertical specialization explain the growth of world trade? *Journal of political Economy*, *111*(1), 52–102.

Online Appendix

1 Calibration

2 Proofs for the nonparametric model

2.1 Firms

2.1.1 Aggregators' Problem

For every sector $i \in \mathscr{N}$, the perfectly competitive aggregator chooses $\left\{y_i,\, (y_{z_i})_{z_i \in [0,1]}\right\}$ to maximize

$$\bar{\pi}_i = p_i y_i - \int p_{z_i} y_{z_i}\, dz_i$$

subject to the CES technology $y_i = \left(y_{z_i}^{\mu_i}\, d\, z_i\right)^{\frac{1}{\mu_i}}$ and taking prices $\left\{p_i, (p_{z_i})_{z_i \in [0,1]}\right\}$ as given.

Taking first order conditions I arrive to the usual Dixit & Stiglitz's (1977) CES demand function

$$y_{z_i} = \left(\frac{p_i}{p_{z_i}}\right)^{\frac{1}{1-\mu_i}} y_i \qquad \forall z_i \in [0,1]\,, \tag{22}$$

from here $\frac{\partial p_{z_i}}{\partial y_{z_i}} = -\left(1-\mu_i\right)\left(\frac{y_i}{y_{z_i}}\right)^{1-\mu_i} \frac{p_i}{y_{z_i}}$ and $p_i = \left(\int p_{z_i}^{\frac{\mu_i}{\mu_i - 1}}\, dz_i\right)^{\frac{\mu_i - 1}{\mu_i}}$.

2.1.2 Monopolistically Competitive Firms' problem

Firm z_i in sector $i \in \mathscr{N}_r$ chooses $\left\{y_{z_i},\, p_{z_i},\, \{\ell_{z_i h}\}_{h \in \mathscr{H}},\, \{x_{z_i j}\}_{j \in \mathscr{N}}\right\}$ to maximize

$$\pi_{z_i} = p_{z_i}\, y_{z_i} - \underbrace{\sum_{h \in \mathscr{H}} w_h\, \ell_{z_i h}}_{= p^{\ell}_{z_i} L_{z_i}} - \underbrace{\sum_{j \in \mathscr{N}} p_j\, x_{z_i j}}_{= p^{x}_{z_i} X_{z_i}}, \tag{23}$$

subject to (22),

$$y_{z_i} = A_i\, Q_i\left(L_{z_i}, X_{z_i})\right), \quad L_{z_i} = A^{\ell}_i\, Q^{\ell}_i\left(\left\{A^{\ell}_{ih}\, \ell_{z_i h}\right\}_{h \in \mathscr{H}}\right), \quad X_{z_i} = A^{x}_i\, Q^{x}_i\left(\left\{A^{x}_{ij}\, x_{z_i j}\right\}_{j \in \mathscr{N}}\right), \tag{24}$$

and taking $\left\{\{w_h\}_{h \in \mathscr{H}},\, \{p_j\}_{j \in \mathscr{N}}\right\}$ as given.

Notice that firm z_i's gross revenue derivative with respect to any variable q is given by

$$\begin{aligned}\frac{\partial\, p_{z_i}\, y_{z_i}}{\partial q} &= \left(p_{z_i} + \frac{\partial p_{z_i}}{\partial y_{z_i}}\, y_{z_i}\right)\frac{\partial y_{z_i}}{\partial q} \\ &= \left(p_{z_i} - (1-\mu_i)\left(\frac{y_{z_i}}{y_i}\right)^{\mu_i - 1} p_i\right)\frac{\partial y_{z_i}}{\partial q} = \mu_i\, p_{z_i}\frac{\partial y_{z_i}}{\partial q}.\end{aligned}$$

Firms z_i's optimality conditions are given by

$$\mu_i\, p_{z_i}\, A_i\, \frac{\partial\, Q_i\,(L_{z_i},\, X_{z_i})}{\partial\, L_{z_i}} = p^{\ell}_{z_i}, \tag{25}$$

$$\mu_i\, p_{z_i}\, A_i\, \frac{\partial\, Q_i\,(L_{z_i},\, X_{z_i})}{\partial\, X_{z_i}} = p^{x}_{z_i}, \tag{26}$$

$$\mu_i\, p_{z_i}\, A_i\, \frac{\partial\, Q_i\,(L_{z_i},\, X_{z_i})}{\partial\, L_{z_i}}\, A^{\ell}_i\, \frac{\partial\, Q^{\ell}_i\left(\left\{A^{\ell}_{ib}\, \ell_{z_i b}\right\}_{b\in\mathscr{H}}\right)}{\partial\, \ell_{z_i h}} = w_h \qquad \forall h \in \mathscr{H} : \partial\, y_{z_i}/\partial\, \ell_{z_i h} > 0, \tag{27}$$

$$\mu_i\, p_{z_i}\, A_i\, \frac{\partial\, Q_i\,(L_{z_i},\, X_{z_i})}{\partial\, X_{z_i}}\, A^{x}_i\, \frac{\partial\, Q^{x}_i\left(\left\{A^{x}_{im}\, x_{z_i m}\right\}_{m\in\mathscr{N}}\right)}{\partial\, x_{z_i j}} = p_j \qquad \forall j \in \mathscr{N} : \partial\, y_{z_i}/\partial\, x_{z_i j} > 0. \tag{28}$$

Representing elasticities with $e\,(a, b) = (\partial a/\partial b)\,(b/a)$ the former first order conditions for firm z_i are also represented by

$$\omega^{\ell}_{z_i} = e\,(y_{z_i},\, L_{z_i}) = \frac{1}{\mu_i}\, \frac{p^{\ell}_{z_i}\, L_{z_i}}{p_{z_i}\, y_{z_i}}, \tag{29}$$

$$\omega^{x}_{z_i} = e\,(y_{z_i},\, X_{z_i}) = \frac{1}{\mu_i}\, \frac{p^{x}_{z_i}\, X_{z_i}}{p_{z_i}\, y_{z_i}}, \tag{30}$$

$$e\,(y_{z_i},\, \ell_{z_i h}) = \frac{1}{\mu_i}\, \frac{w_h\, \ell_{z_i h}}{p_{z_i}\, y_{z_i}} \qquad \forall h \in \mathscr{H} \tag{31}$$

$$e\,(y_{z_i},\, x_{z_i j}) = \frac{1}{\mu_i}\, \frac{p_j\, x_{z_i j}}{p_{z_i}\, y_{z_i}} \qquad \forall j \in \mathscr{N}. \tag{32}$$

Combining equations (25) with (27), and (26) with (28)

$$\alpha_{z_i h} = e\,(L_{z_i},\, \ell_{z_i h}) = \frac{w_h\, \ell_{z_i h}}{p^{\ell}_{z_i}\, L_{z_i}}, \qquad \forall h \in \mathscr{H} \tag{33}$$

$$\omega^{x}_{z_i j} = e\,(X_{z_i},\, x_{z_i j}) = \frac{p_j\, x_{z_i j}}{p^{x}_{z_i}\, X_{z_i}} \qquad \forall j \in \mathscr{N} \tag{34}$$

Additionally, combining (31), (32), and using the implicit function theorem

$$e\,(\ell_{z_i h},\, \ell_{z_i b}) = -\frac{w_b\, \ell_{z_i b}}{w_h\, \ell_{z_i h}} \qquad \forall h, b \in \mathscr{H}, \tag{35}$$

$$e\left(x_{z_i j},\, x_{z_i m}\right) = -\frac{p_m\, x_{z_i m}}{p_j\, x_{z_i j}} \qquad \forall j, m \in \mathscr{N}. \tag{36}$$

Introducing equations (31)-(32) in the cost function

$$\begin{aligned} c_{z_i}\left(\vartheta,\, \rho\right) &= p^{\ell}_{z_i}\, L_{zi} + p^{x}_{z_i}\, x_{zi} = \sum_{h\in\mathscr{H}} w_h\, \ell_{z_i h} + \sum_{j\in\mathscr{N}} p_j\, x_{z_i j} \\ &= \mu_i\, p_{z_i}\, y_{z_i} \left(\sum_{h\in\mathscr{H}} e\left(y_{z_i},\, \ell_{z_i h}\right) + \sum_{j\in\mathscr{N}} e\left(y_{z_i},\, x_{z_i j}\right) \right). \end{aligned} \tag{37}$$

From CRS in $Q_i\left(L_{z_i},\, X_{z_i}\right)$, $Q^{\ell}_i\left(\left\{A^{\ell}_{ih}\, \ell_{z_i h}\right\}_{h\in\mathscr{H}}\right)$, and $Q^{x}_i\left(\left\{A^{x}_{ij}\, x_{z_i j}\right\}_{j\in\mathscr{N}}\right)$

$$\begin{aligned} &\sum_{h\in\mathscr{H}} e\left(y_{z_i},\, \ell_{z_i h}\right) + \sum_{j\in\mathscr{N}} e\left(y_{z_i},\, x_{z_i j}\right) \\ &= e\left(y_{z_i},\, L_{z_i}\right) \sum_{h\in\mathscr{H}} e\left(L_{z_i},\, \ell_{z_i h}\right) + e\left(y_{z_i},\, X_{z_i}\right) \sum_{j\in\mathscr{N}} e\left(X_{z_i},\, x_{z_i j}\right) \\ &= e\left(y_{z_i},\, L_{z_i}\right) + e\left(y_{z_i},\, X_{z_i}\right) = 1, \end{aligned}$$

which implies that in (37) $c_{z_i}\left(\vartheta,\, \rho\right) = \mu_i\, p_{z_i}\, y_{z_i}$, and from here I obtain $\omega^{\ell}_{z_i} = e\left(y_{z_i},\, L_{z_i}\right)$, $\omega^{x}_{z_i} = e\left(y_{z_i},\, X_{z_i}\right)$, $\widetilde{\Omega}^{\ell}_{z_i h} = e\left(y_{z_i},\, \ell_{z_i h}\right)$, and $\widetilde{\Omega}^{x}_{z_i j} = e\left(y_{z_i},\, x_{z_i j}\right)$.

2.2 Households' Problem

Household $h \in \mathscr{H}$ chooses $\left\{\left\{C_{hi}\right\}_{i\in\mathscr{N}},\, L_h\right\}$ to maximize $U_h\left(C_h,\, L_h\right)$ subject to $C_h = Q^{c}_h\left(\left\{C_{hi}\right\}_{i\in\mathscr{N}}\right)$, the budget constraint

$$E_h = p^{c}_h\, C_h = \sum_{i\in\mathscr{N}} p_i\, C_{hi} \leq w_h\, L_h + \Pi_h + T_h, \tag{38}$$

$$\Pi_h = \sum_{i\in\mathscr{N}} \kappa_{ih} \left(\bar{\pi}_i + \int \pi_{z_i}\, dz_i\right), \tag{39}$$

and taking as given

$$\left\{ w_h,\, \left\{ p_i,\, \kappa_{ih},\, \bar{\pi}_i,\, \left(\pi_{z_i}\right)_{z_i\in[0,1]} \right\}_{i\in\mathscr{N}} \right\}.$$

The first order conditions are given by

$$U_{C_h} = \daleth_h\, p^{c}_h, \tag{40}$$

$$U_{L_h} = -\daleth_h\, w_h, \tag{41}$$

$$U_{C_h} \frac{\partial C_h}{\partial C_{hi}} = \daleth_h \, p_i \qquad \forall i \in \mathscr{N} : \, \partial C_h / \partial C_{hi} > 0 \tag{42}$$

where $\daleth_h$ stands for the lagrange multiplier for household h's budget constraint.

Combining (40) with (41), and (40) with (42), the former first order conditions for household h can be represented by

$$\frac{w_h}{p_h^c} U_{C_h} = -U_{L_h}, \tag{43}$$

$$\frac{p_i}{p_h^c} = \frac{\partial C_h}{\partial C_{hi}} \qquad \forall i \in \mathscr{N} : \, \partial C_h / \partial C_{hi} > 0. \tag{44}$$

Using the implicit function theorem, equations (43) and (44) can be represented in terms of elasticities as

$$e\left(C_h, \, L_h\right) = \frac{w_h \, L_h}{p_h^c \, C_h}, \tag{45}$$

$$\beta_{hi} = e\left(C_h, \, C_{hi}\right) = \frac{p_i \, C_{hi}}{p_h^c \, C_h} \qquad \forall i \in \mathscr{N}, \tag{46}$$

$$e\left(C_{hi}, \, C_{hm}\right) + \frac{p_m \, C_{hm}}{p_i \, C_{hi}} = 0 \qquad \forall i, m \in \mathscr{N} : \, \partial C_h / \partial C_{hi} > 0, \tag{47}$$

$$e\left(C_{hi}, \, L_h\right) = \frac{w_h \, L_h}{p_i \, C_{hi}} \qquad \forall i \in \mathscr{N} : \, \partial C_h / \partial C_{hi} > 0. \tag{48}$$

2.3 Government

Government from country $r \in \mathscr{R}$ operates under the following fiscal constraint

$$\sum_{h \in \mathscr{H}_r} T_h = 0. \tag{49}$$

2.4 Proof for Proposition 1

2.4.1 Proof of Necessity

First, using equations (22), (28), and (47), I can obtain the first subset of conditions in Proposition 1

$$\frac{\partial C_h / \partial C_{hj}}{\partial C_h / \partial C_{hi}} = \frac{p_j}{p_i} = \mu_i \left(\frac{y_i}{y_{z_i}}\right)^{1-\mu_i} \frac{\partial y_{z_i}}{\partial x_{z_i j}} \quad \forall i, j \in \mathscr{N}, \, \forall h \in \mathscr{H}, \, \forall z_i \in [0,1], \tag{50}$$
$$\text{such that} \quad \partial C_h / \partial C_{hi} > 0, \quad \partial C_h / \partial C_{hj} > 0, \quad \text{and} \quad \partial y_{z_i} / \partial x_{z_i j} > 0.$$

Notice that in this first subset of equilibrium conditions, household h has to consume both from the sectors i and j, and firms z_i also has to demand intermediate inputs from sector j.

Second, using equations (22), (27), and (48), I can obtain

$$-\frac{w_b}{w_h}\frac{U_{L_h}}{U_{C_{hi}}} = \frac{w_b}{p_i} = \mu_i \left(\frac{y_i}{y_{z_i}}\right)^{1-\mu_i} \frac{\partial\, y_{z_i}}{\partial\, \ell_{z_i b}} \quad \forall i \in \mathscr{N},\ \forall h, b \in \mathscr{H}, \\ \forall z_i \in [0,1]\,, \quad \text{such that} \quad \partial\, C_h/\partial\, C_{hi} > 0, \quad U_{L_h} \neq 0, \quad \text{and} \quad \partial\, y_{z_i}/\partial\, \ell_{z_i b} > 0. \tag{51}$$

Notice that in this second subset of equilibrium conditions, the condition that links the demand from firm z_i for workers of type b and the marginal rate of substitution between the labor supply from households of type h and their consumption of goods form sector i does not require that firm z_i hires workers of type h. What is necessary for this relationship to exist is that firm z_i hires labor from any worker b, and that household h consumes from sector i. Whenever $b \neq h$, the distributional factor-rate-differential wedge w_b/w_h arises.

Finally, the resource constraints

$$y_i = \sum_{h\in\mathscr{H}} C_{hi} + \sum_{j\in\mathscr{N}} \int x_{z_j i}\, dz_j \quad \forall i \in \mathscr{N}, \qquad \text{and} \qquad L_h = \sum_{i\in\mathscr{N}} \int \ell_{z_i h}\, dz_i \quad \forall h \in \mathscr{H}, \tag{52}$$

and the fiscal constraints from equation (49) are necessary conditions for the equilibrium allocation.

2.4.2 Proof of Sufficiency

Now, I am going to prove that for any exogenous set of distortions and equity distribution

$$\left\{\mu_i,\, \{\kappa_{ih}\}_{h\in\mathscr{H}}\right\}_{i\in\mathscr{N}},$$

there exists a strictly positive price system

$$\left\{\left\{(p_{z_i})_{z_i\in[0,1]}\,,\, p_i\right\}_{i\in\mathscr{N}},\, \{w_h\}_{h\in\mathscr{H}}\right\},$$

that implements a specific allocation for firms

$$\left\{\left(y_{z_i},\, \{\ell_{z_i h}\}_{h\in\mathscr{H}}\,,\, \{x_{z_i j}\}_{j\in\mathscr{N}}\right)_{z_i\in[0,1]}\,,\, y_i\right\}_{i\in\mathscr{N}},$$

and a household allocation

$$\left\{\{C_{hi}\}_{i\in\mathscr{N}}\,,\, C_h,\, L_h\right\}_{h\in\mathscr{H}},$$

as an equilibrium.

Let me start by using a normalized price system in which a CRS function defines the global GDP deflator

$$\overline{p}_Y = Q^p\left(\{p_i\}_{i\in\mathscr{N}}\right) = 1. \tag{53}$$

Using equation (27), prices for firm z_i in sector $i \in \mathscr{N}_r$ are given by

$$\begin{aligned} p_{z_i} &= \frac{w_h}{\mu_i}\left(\frac{\partial\, y_{z_i}}{\partial\, \ell_{z_i h}}\right)^{-1} \quad \text{if} \quad \exists h \in \mathscr{H} : \frac{\partial\, y_{z_i}}{\partial\, \ell_{z_i h}} > 0 \quad \text{otherwise} \\ p_{z_i} &= \frac{w_h}{\mu_i}\left(\frac{\partial\, y_{z_i}}{\partial\, x_{z_i \underline{j}}}\right)^{-1}\left(\frac{\partial\, y_{z_{\overline{j}}}}{\partial\, \ell_{\overline{j}h}}\right)^{-1}\prod_{j\in\mathscr{N}_{z_i}}\frac{1}{\mu_j}\left(\frac{y_{z_j}}{y_j}\right)^{1-\mu_j}\prod_{j\in\mathscr{N}_{z_i}\setminus\{\overline{j}\}}\left(\frac{\partial\, y_{z_j}}{\partial\, x_{z_j j+1}}\right)^{-1} \end{aligned} \tag{54}$$

where $\mathscr{N}_{z_i} = \left\{\underline{j}, \underline{j}+1, \cdots, \overline{j}-1, \overline{j}\right\}$ captures a sequence of sectors for which there is sequence of firms that establish a connection between the labor supply from households of type h and the intermediate input demand from firm z_i. What I strictly need for this proof is that $\forall i \in \mathscr{N}$, there $\exists h \in \mathscr{H}$, such that for every firm in sector i, there is some direct or indirect demand of the factor supplied by a worker of type h, and that for every type of worker $h \in \mathscr{H}$, there exists a sector $i \in \mathscr{N}$ that satisfies this condition.

As a consequence, prices for sector $i \in \mathscr{N}$ are given by

$$\begin{aligned} p_i = \frac{w_h}{\mu_i}\Bigg(&\int \mathbb{1}\{\ell_{z_i h} > 0\}\left(\frac{\partial\, y_{z_i}}{\partial\, \ell_{z_i h}}\right)^{\frac{\mu_i}{\mu_i - 1}} d\, z_i \\ &+ \int \mathbb{1}\{\ell_{z_i h} = 0\}\left(\frac{\partial\, y_{z_i}}{\partial\, x_{z_i \underline{j}}}\frac{\partial\, y_{z_{\overline{j}}}}{\partial\, \ell_{z_{\overline{j}} h}}\prod_{j\in\mathscr{N}_{z_i}}\mu_j\left(\frac{y_j}{y_{z_j}}\right)^{1-\mu_j}\prod_{j\in\mathscr{N}_{z_i}\setminus\{\overline{j}\}}\frac{\partial\, y_{z_j}}{\partial\, x_{z_j j+1}}\right)^{\frac{\mu_i}{\mu_i - 1}} d\, z_i\Bigg)^{\frac{1-\mu_i}{\mu_i}}. \end{aligned} \tag{55}$$

From equation (53) wages for households of type $h \in \mathscr{H}_q$ are given by

$$\begin{aligned} w_h = Q^p\Bigg(\Bigg\{\Bigg\{\frac{1}{\mu_i}\Bigg(&\int \mathbb{1}\{\ell_{z_i h} > 0\}\left(\frac{\partial\, y_{z_i}}{\partial\, \ell_{z_i h}}\right)^{\frac{\mu_i}{\mu_i - 1}} d\, z_i \\ &+ \int \mathbb{1}\{\ell_{z_i h} = 0\}\left(\frac{\partial\, y_{z_i}}{\partial\, x_{z_i \underline{j}}}\frac{\partial\, y_{z_{\overline{j}}}}{\partial\, \ell_{z_{\overline{j}} h}}\prod_{j\in\mathscr{N}_{z_i}}\mu_j\left(\frac{y_j}{y_{z_j}}\right)^{1-\mu_j}\prod_{j\in\mathscr{N}_{z_i}\setminus\{\overline{j}\}}\frac{\partial\, y_{z_j}}{\partial\, x_{z_j j+1}}\right)^{\frac{\mu_i}{\mu_i - 1}} d\, z_i\Bigg)^{\frac{1-\mu_i}{\mu_i}}\Bigg\}_{i\in\mathscr{N}_r}\Bigg\}_{r\in\mathscr{R}}\Bigg)^{-1}. \end{aligned} \tag{56}$$

Notice that prices and wages are strictly positive because the marginal productivities of factors and intermediate inputs have to be strictly positive when there is some demand.

Now, I need to prove that starting from the set of equilibrium conditions represented in equations (50), (51), and (52), and under the system of prices represented in equations (55) and (56), the optimality conditions for firms and households hold.

To obtain equations (47) and (48), assume that firms in sector i directly or indirectly demand workers of type h, and firms in sector j directly or indirectly demand workers of type b. This assumption is made without loss of generality as it holds for any combination of pairs $i, j \in \mathscr{N}$ and $h, b \in \mathscr{H}$. Introducing equations (50) and (51) in (55)

$$p_i = \frac{w_h}{\mu_i} \left(\left(\frac{w_b}{w_h} \mu_i \frac{U_{C_{bi}}}{U_{L_b}} \right)^{\frac{\mu_i}{1-\mu_i}} \int \left(\frac{y_i}{y_{z_i}} \right)^{\mu_i} d\, z_i \right)^{\frac{1-\mu_i}{\mu_i}} = -w_b \frac{U_{C_{bi}}}{U_{L_b}},$$

$$p_j = -w_b \frac{U_{C_{bj}}}{U_{L_b}}.$$

This proofs (48). Dividing these conditions, I arrive to $\frac{U_{C_{bj}}}{U_{C_{bi}}} = \frac{p_j}{p_i}$, which is (47).

Equation (45) comes from multiplying equation (48) by C_{bi}, adding up over all sectors, using the assumption that $Q^c\left(\{C_{bi}\}_{i \in \mathscr{N}}\right)$ is CRS in conjunction with Euler's homogeneous function theorem, and the implicit function theorem

$$w_b\, U_{C_b} \underbrace{\sum_{i \in \mathscr{N}} C_{bi} \frac{\partial\, C_b}{\partial\, C_{bi}}}_{= C_b} = -U_{L_b} \underbrace{\sum_{i \in \mathscr{N}} p_i\, C_{bi}}_{= p_b^c\, C_b},$$

this implies that $\frac{w_b}{p_b^c} = -\frac{U_{L_b}}{U_{C_b}}$, which is equation (45).

Equation (46) comes from dividing equation (45) by equation (48)

$$\frac{p_i}{p_b^c} = \frac{\partial\, C_b}{\partial\, C_{bi}}.$$

Now for firms, I obtain equation (32) from equation (50), using the implicit function theorem, and introducing equations (22) and (47)

$$\underbrace{\frac{p_i}{p_j} \frac{\partial\, C_b / \partial\, C_{bj}}{\partial\, C_b / \partial\, C_{bi}}}_{= \xi_{rq}} = \mu_i \frac{p_i}{p_j} \left(\frac{y_i}{y_{z_i}} \right)^{1-\mu_i} \frac{\partial\, y_{z_i}}{\partial\, x_{z_i j}}$$

$$\frac{\partial\, y_{z_i}}{\partial\, x_{z_i j}} = \frac{1}{\mu_i} \frac{p_j}{p_{z_i}} \qquad \forall i, j \in \mathscr{N}_r, \quad \text{and} \quad \forall z_i \in [0,1] : \frac{\partial\, y_{z_i}}{\partial\, x_{z_i j}} > 0.$$

Equation (30) comes from adding up equation (32) over all sectors, and using the assumption that $Q_i^x\left(\left\{A_{ij}^x\, x_{z_i j}\right\}_{j\in\mathscr{N}}\right)$ is CRS in conjunction with Euler's homogeneous function theorem

$$\mu_i\, p_{z_i}\, \frac{\partial\, y_{z_i}}{\partial X_{z_i}}\, A_i^x \underbrace{\sum_{j\in\mathscr{N}} x_{z_i j}\, \frac{\partial\, Q_i^x\left(\left\{A_{ij}^x\, x_{z_i j}\right\}_{j\in\mathscr{N}}\right)}{\partial\, x_{z_i j}}}_{=\,Q_i^x\left(\left\{A_{ij}^x\, x_{z_i j}\right\}_{j\in\mathscr{N}}\right)} = \underbrace{\sum_{j\in\mathscr{N}} p_j\, x_{z_i j}}_{=\,p_{z_i}^x\, X_{z_i}}$$

$$\frac{\partial\, y_{z_i}}{\partial\, X_{z_i}} = \frac{1}{\mu_i}\, \frac{p_{z_i}^x}{p_{z_i}} \qquad \forall i \in \mathscr{N} \quad \text{and} \quad \forall z_i \in [0,1] : \frac{\partial\, y_{z_i}}{\partial\, X_{z_i}} > 0.$$

Equation (31) comes from introducing equations (22) and (48) in equation (51)

$$\underbrace{-\frac{p_i}{w_b}\, \frac{U_{L_b}}{U_{C_{bi}}}}_{=\,\xi_{ru}} = \mu_i\, \frac{p_i}{w_h} \left(\frac{y_i}{y_{z_i}}\right)^{1-\mu_i} \frac{\partial\, y_{z_i}}{\partial\, \ell_{z_i h}}$$

$$\frac{\partial\, y_{z_i}}{\partial\, \ell_{z_i h}} = \frac{1}{\mu_i}\, \frac{w_h}{p_{z_i}} \qquad \forall i \in \mathscr{N}, \quad \forall h \in \mathscr{H}, \quad \text{and} \quad \forall z_i \in [0,1] : \frac{\partial\, y_{z_i}}{\partial\, \ell_{z_i h}} > 0.$$

Equation (29) comes from adding up equations (31) over all households, and using the assumption that $Q_i^l\left(\left\{A_{ih}^\ell\, \ell_{z_i h}\right\}_{h\in\mathscr{H}}\right)$ is CRS in conjunction with Euler's homogeneous function theorem

$$\mu_i\, p_{z_i}\, \frac{\partial\, y_{z_i}}{\partial L_{z_i}}\, A_i^\ell \underbrace{\sum_{h\in\mathscr{H}} \ell_{z_i h}\, \frac{\partial\, Q_i^\ell\left(\left\{A_{ih}^\ell\, \ell_{z_i b}\right\}_{b\in\mathscr{H}}\right)}{\partial\, \ell_{z_i h}}}_{=\,Q_i^\ell\left(\left\{A_{ih}^\ell\, \ell_{z_i h}\right\}_{h\in\mathscr{H}}\right)} = \underbrace{\sum_{h\in\mathscr{H}} w_h\, \ell_{z_i h}}_{=\,p_{z_i}^\ell\, L_{z_i}}$$

$$\frac{\partial\, y_{z_i}}{\partial L_{z_i}} = \frac{1}{\mu_i}\, \frac{p_{z_i}^\ell}{p_{z_i}} \qquad \forall i \in \mathscr{N} \quad \text{and} \quad z_i \in [0,1] : \frac{\partial\, y_{z_i}}{\partial\, L_{z_i}} > 0.$$

What remains to be proven is is that households' budget constraints hold. Adding up equation (38), and introducing equation (39)

$$\sum_{h\in\mathscr{H}} \sum_{i\in\mathscr{N}} p_i\, C_{hi} = \sum_{h\in\mathscr{H}} \left(w_h\, L_h + \sum_{i\in\mathscr{N}} \kappa_{ih} \left(\bar{\pi}_i + \int \pi_{z_i}\, dz_i\right) + T_h\right).$$

Introducing zero-profit condition on aggregator firms ($\bar{\pi}_i = 0 \quad \forall i \in \mathscr{N}$), equations (23) and (49), and rearranging terms

$$\sum_{h\in\mathscr{H}} \sum_{i\in\mathscr{N}} p_i\, C_{hi} = \sum_{h\in\mathscr{H}} w_h\, L_h + \sum_{h\in\mathscr{H}} \sum_{i\in\mathscr{N}} \kappa_{ih} \int \left(p_{z_i}\, y_{z_i} - \sum_{b\in\mathscr{H}} w_b\, \ell_{z_i b} - \sum_{j\in\mathscr{N}} p_j\, x_{z_i j}\right) dz_i$$

$$\sum_{h\in\mathscr{H}}\sum_{i\in\mathscr{N}} p_i\, C_{hi} = \sum_{h\in\mathscr{H}} w_h\, L_h + \sum_{i\in\mathscr{N}} \int \left(p_{z_i}\, y_{z_i} - \sum_{b\in\mathscr{H}} w_b\, \ell_{z_i b} - \sum_{j\in\mathscr{N}} p_j\, x_{z_i j} \right) dz_i \underbrace{\sum_{h\in\mathscr{H}} \kappa_{ih}}_{=1}$$

$$0 = \sum_{i\in\mathscr{N}} \left(\int p_{z_i}\, y_i\, d\, z_i - p_i \sum_{h\in\mathscr{H}} C_{hi} - p_i \sum_{j\in\mathscr{N}} \int x_{z_j i}\, d\, z_j \right) + \sum_{h\in\mathscr{H}} w_h \left(L_h - \sum_{q\in\mathscr{R}} \sum_{i\in\mathscr{N}_q} \int \ell_{z_i h} \right)$$

From zero profits for aggregators $p_i y_i = \int p_{z_i} y_{z_i}$, and using equations (52), the households' budget constraints holds

$$0 = \sum_{i\in\mathscr{N}} p_i \underbrace{\left(y_i - \sum_{h\in\mathscr{H}} C_{hi} - \sum_{j\in\mathscr{N}} \int x_{z_j i}\, dz_j \right)}_{=0} + \sum_{h\in\mathscr{H}} w_h \underbrace{\left(L_h - \sum_{i\in\mathscr{N}} \int \ell_{z_i h}\, dz_i \right)}_{=0}.$$

2.5 Equilibrium Centralities from Subsection 3.2

2.5.1 Goods Market Equilibrium Conditions

Introducing (30), (32), (34), and (46) in the goods market resource constraint (52) for sector $i \in \mathscr{N}$

$$S_i = \sum_{h\in\mathscr{H}} p_i\, C_{hi} + \sum_{j\in\mathscr{N}} \int p_i\, x_{z_j i}\, dz_j = \sum_{h\in\mathscr{H}} \beta_{hi}\, E_h + \sum_{j\in\mathscr{N}} \mu_j \int \omega^x_{z_j}\, \omega_{z_j i}\, p_{z_j}\, y_{z_j}\, dz_j.$$

Imposing symmetry in the decision of monopolistically competitive firms within the same sector

$$S_i = \sum_{h\in\mathscr{H}} \beta_{hi}\, E_h + \sum_{j\in\mathscr{N}} \Omega^x_{ji}\, S_j, \tag{57}$$

where $\Omega^x_{ij} \equiv \mu_i\, \widetilde{\Omega}^x_{ij}$ and $\widetilde{\Omega}^x_{ij} = \omega^\ell_i\, \omega_{ij}$.

In matrix form, this equation is represented by

$$S = \beta'\, E + \widetilde{\Omega}'_x\, diag\,(\mu)\; S, \tag{58}$$

where $S \equiv [\, S_1, \cdots, S_N]'$, $E \equiv [\, E_1, \cdots, E_H]'$, $\mu \equiv [\, \mu_1, \cdots, \mu_N]'$, and the matrices

$$\beta \equiv \begin{pmatrix} \beta_{11} & \cdots & \beta_{1N} \\ \vdots & \ddots & \vdots \\ \beta_{H1} & \cdots & \beta_{HN} \end{pmatrix}, \quad \Omega_x = \begin{pmatrix} \Omega^x_{11} & \cdots & \Omega^x_{1N} \\ \vdots & \ddots & \vdots \\ \Omega^x_{N1} & \cdots & \Omega^x_{NN} \end{pmatrix},$$

$$\Omega_x \equiv diag\,(\mu)\,\widetilde{\Omega}_x.$$

By dividing equation (57) by global nominal GDP, I arrive to the following equation that relates the revenue-based Domar weights and the absorption shares

$$\left(I_N - \widetilde{\Omega}_x'\,diag\,(\mu)\right)\lambda = \beta'\chi,$$

$$\lambda = \mathscr{B}'\,\chi, \tag{59}$$

where $\lambda \equiv [\,\lambda_1, \cdots, \lambda_N]'$, $\chi \equiv [\,\chi_1, \cdots, \chi_H]'$, $\Psi_x \equiv (I_N - \Omega_x)^{-1}$, and $\mathscr{B} \equiv \beta\,\Psi_x$. In equilibrium, λ_i captures the share of global expenditure that reaches sector i.

Let me define the cost-based Domar weights

$$\widetilde{\lambda} \equiv \widetilde{\mathscr{B}}'\,\chi, \tag{60}$$

with $\widetilde{\mathscr{B}} \equiv \beta\,\widetilde{\Psi}_x \equiv \beta$ and $\widetilde{\Psi}_x \equiv \left(I_N - \widetilde{\Omega}_x\right)^{-1}$.

To understand the cost-based Domar weights, notice that

$$\widetilde{\lambda}_i \equiv \sum_{h\in\mathscr{H}} \beta_{hi}\,\chi_h + \sum_{j\in\mathscr{N}} \widetilde{\Omega}^x_{ji}\,\widetilde{\lambda}_j = \sum_{h\in\mathscr{H}} \widetilde{\mathscr{B}}_{hi}\,\chi_h.$$

Remember that in equilibrium, β_{hi} represents the expenditure share from households of type h that is used to acquire goods from sector i, and $\widetilde{\Omega}^x_{ji}$ captures the cost share in sector j of intermediate goods supplied by sector i. Hence, $\widetilde{\mathscr{B}}_{hi}$ represent the share of expenditure for households of type h that can be traced back to the supply of goods from sector i. For this reason, for a specific consumption expenditure distribution χ, $\widetilde{\lambda}_i$ captures the aggregate value-added share that passes through sector i. Notice that $\omega_\ell'\,\widetilde{\lambda} = \mathbb{1}_N'\left(I_N - \widetilde{\Omega}_x'\right)\widetilde{\Psi}_x'\,\beta'\,\chi = 1$, and for this reason $\omega^\ell_i\,\widetilde{\lambda}_i$ is the aggregate share of value-added from sector generated by workers in sector i.

Finally, I am going to prove that the value-added that passes through a sector is greater than or equal to its revenue, i.e., that $\widetilde{\lambda}_i \geq \lambda_i$ holds $\forall i \in \mathscr{N}$. Let me start with

$$\widetilde{\Psi}_x - \Psi_x = \widetilde{\Psi}_x - \Psi_x = \sum_{q=1}^{\infty}\left(\widetilde{\Omega}^q_x - \Omega^q_x\right).$$

Notice that $\widetilde{\Omega}_x - \Omega_x = \left(\widetilde{\Omega}_x - diag\,(\mu)\,\widetilde{\Omega}_x\right) \succcurlyeq 0_N\,0_N'$, because $\mu_i \in (0,1]$ and $\widetilde{\Omega}^x_{ij} \geq 0$ ($A \succcurlyeq B$ means that matrix A is elementwise greater than or equal than matrix B). Now, from induction, for $q > 1$ assume that $\widetilde{\Omega}_x^{q-1} - \Omega_x^{q-1} \succcurlyeq 0_N\, 0_N'$, then

$$\begin{aligned}
\widetilde{\Omega}_x^q - \Omega_x^q &= \left(\widetilde{\Omega}_x^q - \Omega_x^{q-1} diag\,(\mu)\, \widetilde{\Omega}_x\right) \\
&= \left(\widetilde{\Omega}_x^q - \Omega_x^{q-1}\widetilde{\Omega}_x + \Omega_x^{q-1}\left(\widetilde{\Omega}_x - diag\,(\mu)\,\widetilde{\Omega}_x\right)\right) \\
&= \left(\widetilde{\Omega}_x^{q-1}\left(I_N - \widetilde{\Omega}_x\right) + \Omega_x^{q-1}\left(\widetilde{\Omega}_x - diag\,(\mu)\,\widetilde{\Omega}_x\right)\right) \succcurlyeq 0_N\, 0_N'.
\end{aligned}$$

Therefore $\widetilde{\Psi}_x \succcurlyeq \Psi_x$. As a consequence $\widetilde{\mathscr{B}} - \mathscr{B} = \beta\left(\widetilde{\Psi}_x - \Psi_x\right) \succcurlyeq 0_H\, 0_N'$ because $\widetilde{\lambda} - \lambda = \left(\widetilde{\mathscr{B}} - \mathscr{B}\right)' \chi \succcurlyeq 0_N$.

2.5.2 Labor Market Equilibrium Conditions

Introducing (29), (31), and (33) in the labor market clearing condition (52) for household $h \in \mathscr{H}$

$$J_h = w_h\, L_h = \sum_{i\in\mathscr{N}} \int w_h\, \ell_{z_i h}\, dz_i = \sum_{i\in\mathscr{N}} \mu_i \int \omega_{z_i}^{\ell}\, \alpha_{z_i h}\, p_{z_i}\, y_{z_i}\, dz_i.$$

Imposing symmetry in the decision of monopolistically competitive firms within the same sector

$$J_h = \sum_{i\in\mathscr{N}} \mu_i\, \widetilde{\Omega}_{ih}^{\ell}\, S_i, \tag{61}$$

where $\Omega_{ih}^{\ell} = \mu_i \widetilde{\Omega}_{ih}^{\ell}$ and $\widetilde{\Omega}_{ih}^{\ell} \equiv \omega_i^{\ell}\, \alpha_{ih}$.

In matrix form, these equations are represented by

$$J = \widetilde{\Omega}_\ell'\, diag\,(\mu)\; S = \Omega_\ell'\, S, \tag{62}$$

where the matrices are given by

$$\Omega_\ell \equiv \begin{pmatrix} \Omega_{11}^{\ell} & \cdots & \Omega_{1H}^{\ell} \\ \vdots & \ddots & \vdots \\ \Omega_{N1}^{\ell} & \cdots & \Omega_{NH}^{\ell} \end{pmatrix}, \quad \Omega_\ell \equiv diag\,(\mu)\; \widetilde{\Omega}_\ell$$

and $J \equiv [\, J_1, \cdots, J_H]'$.

By dividing equation (61) by global nominal GDP, I arrive at the following equation that relates the labor income shares and the revenue-based Domar weights

$$\Lambda = \Omega_\ell'\, \lambda, \tag{63}$$

where $\Lambda \equiv [\Lambda_1, \cdots, \Lambda_H]'$.

I define the cost-based factor Domar weights as

$$\widetilde{\Lambda} \equiv \widetilde{\Omega}_\ell' \, \widetilde{\lambda}, \tag{64}$$

where $\mathbb{1}'_H \, \widetilde{\Lambda} = \mathbb{1}'_H \, \alpha' \, diag\,(\omega_\ell) \; \widetilde{\lambda} = \omega_\ell' \, \widetilde{\lambda} = \sum_{i \in \mathscr{N}} \omega_i^\ell \, \widetilde{\lambda}_i = 1$.

Notice that $\widetilde{\Lambda} \succcurlyeq \Lambda$ because

$$\begin{aligned} \widetilde{\Lambda} - \Lambda &= \widetilde{\Omega}_\ell' \, \widetilde{\lambda} - \Omega_\ell' \, \lambda \\ &= \underbrace{\widetilde{\Omega}_\ell'}_{\succcurlyeq 0_H 0_N'} \underbrace{\left(\widetilde{\lambda} - \lambda\right)}_{\succcurlyeq 0_N} + \underbrace{\left(\widetilde{\Omega}_\ell - diag\,(\mu) \; \widetilde{\Omega}_\ell\right)'}_{\succcurlyeq 0_N 0_N'} \lambda. \end{aligned}$$

$\widetilde{\Omega}_\ell \succcurlyeq \Omega_\ell$ holds due to $\mu_i \in (0, 1]$ and $\widetilde{\Omega}_{ih}^\ell \geq 0$.

The firm-to-worker and worker-to-firm centrality matrices are respectively given by

$$\Psi_\ell = \Psi_x \, \Omega_\ell, \qquad\qquad \widetilde{\Psi}_\ell = \widetilde{\Psi}_x \, \widetilde{\Omega}_\ell, \tag{65}$$

where $\widetilde{\Psi}_\ell \, \mathbb{1}_H = \widetilde{\Psi}_x \, \widetilde{\Omega}_\ell \, \mathbb{1}_H = \widetilde{\Psi}_x \, \omega_\ell = \widetilde{\Psi}_x \left(I_N - \widetilde{\Omega}_x\right) \mathbb{1}_N = \mathbb{1}_N$. Additionally $\widetilde{\Psi}_\ell \succcurlyeq \Psi_\ell$ because

$$\widetilde{\Psi}_\ell - \Psi_\ell = \underbrace{\left(\widetilde{\Psi}_x - \Psi_x\right)}_{\succcurlyeq 0_N 0_N'} \underbrace{\widetilde{\Omega}_\ell}_{\succcurlyeq 0_N 0_H'} + \underbrace{\Psi_x}_{\succcurlyeq 0_N 0_N'} \underbrace{\left(\widetilde{\Omega}_\ell - \Omega_\ell\right)}_{\succcurlyeq 0_N 0_N'}.$$

Similarly, the consumer-to-worker and worker-to-consumer centrality matrices are respectively given by

$$\mathscr{C} = \mathscr{B} \, \Omega_\ell, \qquad\qquad \widetilde{\mathscr{C}} = \widetilde{\mathscr{B}} \, \widetilde{\Omega}_\ell, \tag{66}$$

where $\widetilde{\mathscr{C}} \, \mathbb{1}_H = \widetilde{\mathscr{B}} \, \widetilde{\Omega}_\ell \, \mathbb{1}_H = \beta \, \widetilde{\Psi}_x \, \omega_\ell = \beta \, \widetilde{\Psi}_x \left(I_N - \widetilde{\Omega}_x\right) \mathbb{1}_N = \mathbb{1}_H$, $\widetilde{\mathscr{C}}' \, \chi = \widetilde{\Omega}_\ell' \, \widetilde{\mathscr{B}}' \, \chi = \widetilde{\Omega}_\ell' \, \widetilde{\lambda} = \widetilde{\Lambda}$, $\mathscr{C}' \, \chi = \Omega_\ell' \, \mathscr{B}' \chi = \Omega_\ell' \, \lambda = \Lambda$, and $\widetilde{\mathscr{C}} \succcurlyeq \mathscr{C}$ because

$$\widetilde{\mathscr{C}} - \mathscr{C} = \underbrace{\left(\widetilde{\mathscr{B}} - \mathscr{B}\right)}_{\succcurlyeq 0_H 0_N'} \underbrace{\widetilde{\Omega}_\ell}_{\succcurlyeq 0_N 0_H'} + \underbrace{\mathscr{B}}_{\succcurlyeq 0_H 0_N'} \underbrace{\left(\widetilde{\Omega}_\ell - \Omega_\ell\right)}_{\succcurlyeq 0_N 0_N'}.$$

2.5.3 Labor Wedges

From equations (30), (34), and (46), and imposing symmetry in the decision of monopolistically competitive firms within the same sector

$$x_{ji} = \mu_j \, \omega_j^x \, \omega_{ji} \, y_j \frac{\beta_{hj}}{\beta_{hi}} \frac{C_{hi}}{C_{hj}} \qquad \forall h \in \mathscr{H} \quad \text{and} \quad \forall i, j \in \mathscr{N}$$

From equation (52), the goods market resource constraint for goods produced firms in sector i in terms of household h's consumption is given by

$$y_i = \sum_{b \in \mathscr{H}} C_{bi} + \frac{C_{hi}}{\beta_{hi}} \sum_{j \in \mathscr{N}} \mu_j \, \omega_j^x \, \omega_{ji} \, y_j \frac{\beta_{hj}}{C_{hj}}$$

In matrix representation, this equation is given by

$$y = C' \mathbb{1}_H + diag\left(\left(\beta^{\circ -1} \circ C\right)' o_H\left(h\right) \right) \Omega_x' diag\left(\left(\beta^{\circ -1} \circ C\right)' o_H\left(h\right) \right)^{-1} y,$$

$$y = \left[I_N - diag\left(\left(\beta^{\circ -1} \circ C\right)' o_H\left(h\right) \right) \Omega_x' diag\left(\left(\beta^{\circ -1} \circ C\right)' o_H\left(h\right) \right)^{-1} \right]^{-1} C' \, \mathbb{1}_H,$$

$$y = diag\left(\left(\beta^{\circ -1} \circ C\right)' o_H\left(h\right) \right) \left[I_N - \Omega_x' \right]^{-1} diag\left(\left(\beta^{\circ -1} \circ C\right)' o_H\left(h\right) \right)^{-1} C' \, \mathbb{1}_H,$$

$$diag\left(\left(\beta^{\circ -1} \circ C\right)' o_H\left(h\right) \right)^{-1} y = \Psi_x' diag\left(\left(\beta^{\circ -1} \circ C\right)' o_H\left(h\right) \right)^{-1} C' \, \mathbb{1}_H,$$

where $\circ$ stands for the Hadamard product, $^\circ$ for the Hadamard power, and $o_H\left(h\right)$ for a vector of zeros with size H that has a one in position h.

Notice from equation (46) that $\beta_{hi} \frac{\chi_h}{C_{hi}} = p_i = \beta_{bi} \frac{\chi_b}{C_{bi}}$, and as a consequence

$$diag\left(\left(\beta^{\circ -1} \circ C\right)' o_H\left(h\right) \right)^{-1} C' \, \mathbb{1}_H = \begin{pmatrix} \sum_{b \in \mathscr{H}} \beta_{h1} \frac{C_{b1}}{C_{h1}} \\ \vdots \\ \sum_{b \in \mathscr{H}} \beta_{hN} \frac{C_{bF}}{C_{hF}} \end{pmatrix} = \chi_h^{-1} \beta' \, \chi.$$

Then

$$diag\left(\left(\beta^{\circ -1} \circ C\right)' o_H\left(h\right) \right)^{-1} y = \chi_h^{-1} \, \Psi_x' \, \beta' \, \chi. \tag{67}$$

Now, from equations (29), (33), (46), and (48), and imposing symmetry in the decision of monopolistically competitive firms within the same sector

$$\ell_{ih} = -\frac{U_{C_h}}{U_{L_h}} \mu_i \, \omega_i^{\ell} \, \alpha_{ih} \, y_i \, \beta_{hi} \frac{C_h}{C_{hi}} \qquad \forall h \in \mathscr{H} \quad \text{and} \quad \forall i \in \mathscr{N}.$$

In matrix representation, these conditions are portrayed by

$$\ell_h = -\frac{U_{C_h}}{U_{L_h}} C_h \, diag\left(\Omega_\ell \, o_H(h)\right) diag\left(\left(\beta^{\circ -1} \circ C\right)' o_H(h)\right)^{-1} y.$$

Adding up, the labor market equilibrium from equation (52) in terms of first-order conditions is given by

$$L_h = -\frac{U_{C_h}}{U_{L_h}} C_h \underbrace{\mathbb{1}_N' \, diag\left(\Omega_\ell \, o_H(h)\right) diag\left(\left(\beta^{\circ -1} \circ C\right)' o_H(h)\right)^{-1} y}_{=\Gamma_h}.$$

Consequently, equilibrium labor supply is characterized by

$$L_h + \Gamma_h \frac{U_{C_h}}{U_{L_h}} C_h = 0. \tag{68}$$

Taking equation (67)

$$\begin{aligned}
\Gamma_h &= \chi_h^{-1} \, o_H(h)' \mathscr{C}' \chi \\
&= \chi_h^{-1} \, \mathbb{1}_N' \, diag\left(\Omega_\ell \, o_H(h)\right) \Psi_x' \, \beta' \chi \\
&= \chi_h^{-1} \, \mathbb{1}_N' \, diag\left(\widetilde{\Omega}_\ell \, o_H(h)\right) diag(\mu) \left(I_N - \widetilde{\Omega}_x' \, diag(\mu)\right)^{-1} \beta' \chi \\
&= \chi_h^{-1} \, \mathbb{1}_N' \, diag\left(\widetilde{\Omega}_\ell \, o_H(h)\right) \left(diag(\mu)^{-1} - \widetilde{\Omega}_x'\right)^{-1} \beta' \chi.
\end{aligned} \tag{69}$$

Finally, using equations (59) and (63), in the steady state is given by

$$\begin{aligned}
\Gamma_h &= \chi_h^{-1} \, \mathbb{1}_N' \, diag\left(\Omega_\ell \, o_H(h)\right) \Psi_x' \, \beta' \chi = \chi_h^{-1} \, \mathbb{1}_N' \, diag\left(\Omega_\ell \, o_H(h)\right) \lambda \\
&= \chi_h^{-1} \sum_{i \in \mathscr{N}} \Omega_{ih}^{\ell} \sum_{j \in \mathscr{N}} \psi_{ji}^{x} \sum_{b \in \mathscr{H}} \beta_{bj} \, \chi_b = \chi_h^{-1} \sum_{i \in \mathscr{N}} \Omega_{ih}^{\ell} \, \lambda_i = \frac{\Lambda_h}{\chi_h} \leq 1.
\end{aligned} \tag{70}$$

2.5.4 Household Budget Constraint Equilibrium Conditions

Introducing equations (29) and (30) in the profit equation (23)

$$\pi_{z_i} = (1 - \mu_i) \, p_{z_i} \, y_{z_i}. \tag{71}$$

Introducing (29), (31), (33), (52), and (71) in the budget constraint for household $h \in \mathscr{H}_r$ (38)

$$E_h = \sum_{i \in \mathscr{N}} \int \left(\mu_i \, \omega^{\ell}_{z_i} \, \alpha_{z_i h} + \kappa_{ih} \, (1 - \mu_i)\right) p_{z_i} \, y_{z_i} \, dz_i + T_h. \tag{72}$$

Imposing symmetry in the decision of monopolistically competitive firms within the same sector

$$E_h = \sum_{i \in \mathscr{N}} \left(\mu_i \, \widetilde{\Omega}^{\ell}_{ih} + \kappa_{ih} \, (1 - \mu_i)\right) S_i + T_h. \tag{73}$$

In matrix form, these equations are represented by

$$\chi = (\Omega'_\ell + \Omega'_\pi) \, \lambda + \mathbb{T}, \tag{74}$$

where $\mathbb{T} = [\mathbb{T}_1, \cdots, \mathbb{T}_H]'$, $\mathbb{T}_h = T_h / GDP$,

$$\Omega_\pi = (I_N - diag\,(\mu))\, \kappa, \qquad \text{and} \qquad \kappa \equiv \begin{pmatrix} \kappa_{11} & \cdots & \kappa_{1H} \\ \vdots & \ddots & \vdots \\ \kappa_{N1} & \cdots & \kappa_{NH} \end{pmatrix}.$$

2.5.5 Nominal GDP

To define the nominal GDP for country $r \in \mathscr{R}$, I start by aggregating the good market clearing condition from equation (52) for the subset of sectors $\mathscr{N}_r$ that produce in this country:

$$\sum_{i \in \mathscr{N}_r} S_i = \sum_{i \in \mathscr{N}_r} \left(\sum_{h \in \mathscr{H}} p_i C_{hi} + \sum_{j \in \mathscr{N}} p_i \int x_{z_j i} \, dz_j \right) = \underbrace{\sum_{i \in \mathscr{N}_r} p_i \sum_{h \in \mathscr{H}_r} C_{hi}}_{= E^{dom}_r}$$
$$+ \underbrace{\sum_{i \in \mathscr{N}_r} \left(\sum_{h \notin \mathscr{H}_r} p_i C_{hi} + \sum_{j \notin \mathscr{N}_r} p_i \int x_{z_j i} \, dz_j \right)}_{= Exp_r} + \sum_{i \in \mathscr{N}_r} \sum_{j \in \mathscr{N}_r} p_i \int x_{z_j i} \, dz_j.$$

Now, nominal imports Imp_r are given

$$Imp_r = \underbrace{\sum_{q \in \mathscr{R} \setminus r} \sum_{j \in \mathscr{N}_q} p_j \sum_{h \in \mathscr{H}_r} C_{hj}}_{= E^{for}_r} + \sum_{q \in \mathscr{R} \setminus r} \sum_{j \in \mathscr{N}_q} p_j \sum_{i \in \mathscr{N}_r} \int x_{z_i j} \, dz_i.$$

Then

$$
\begin{aligned}
GDP_r &\equiv E_r^{dom} + E_r^{for} + Exp_r - Imp_r \\
&= \sum_{i\in\mathscr{N}_r}\left(S_i - \sum_{j\in\mathscr{N}_r} p_i \int x_{z_j i}\, dz_j\right) - \sum_{q\in\mathscr{R}\setminus r}\sum_{j\in\mathscr{N}_q} p_j \sum_{i\in\mathscr{N}_r}\int x_{z_i j}\, dz_i \\
&= \sum_{i\in\mathscr{N}_r}\left(S_i - \sum_{j\in\mathscr{N}} p_j \int x_{z_i j}\, dz_i\right),
\end{aligned}
$$

using equations (30), (32), (34), and imposing symmetry in the decision of monopolistically competitive firms within the same sector

$$
GDP_r = \sum_{i\in\mathscr{N}_r}\left(1 - \sum_{j\in\mathscr{N}} \Omega_{ij}^x\right) S_i = \sum_{i\in\mathscr{N}_r} \left(1-\mu_i\,\omega_i^x\right) S_i. \tag{75}
$$

This coincided with the total value-added generated by firms located in country r.

$$
\begin{aligned}
GDP_r &= \sum_{i\in\mathscr{N}_r}\left(\left(\omega_i^\ell + \omega_i^x\right) S_i - \sum_{j\in\mathscr{N}} \Omega_{ij}^x S_i\right) \\
&= \sum_{i\in\mathscr{N}_r}\left(\sum_{h\in\mathscr{H}} w_h\,\ell_{ih} - \mu_i \frac{\sum_{h\in\mathscr{H}} w_h\,\ell_{ih}}{\mu_i\, S_i} S_i + \left(1-\mu_i\sum_{j\in\mathscr{N}}\widetilde{\Omega}_{ij}^x\right) S_i\right) \\
&= \sum_{i\in\mathscr{N}_r}\left(\sum_{h\in\mathscr{H}} w_h\,\ell_{ih} + \left(1-\mu_i\right) S_i\right).
\end{aligned} \tag{76}
$$

In matrix form, country-level GDP is represented by

$$
\overline{GDP} = F_{\mathscr{N}}' \;\left(S - diag\left(S\right)\,\Omega_x\,\mathbb{1}_N\right),
$$

where $F_{\mathscr{N}}$ is a $N\times R$ matrix of zeros than in its column r contains one in the positions that correspond to the sectors that produce in country r.

Additionally, global nominal GDP is given by

$$
GDP = \mathbb{1}_R'\,\overline{GDP} = \sum_{i\in\mathscr{N}}\left(1-\sum_{j\in\mathscr{N}}\Omega_{ij}^x\right) S_i = \sum_{i\in\mathscr{N}} \left(1-\mu_i,\omega_i^x\right) S_i.
$$

Notice that this definition of country level GDP differs from the gross national income GNI_r given by

$$
GNI_r \equiv \sum_{h\in\mathscr{H}_r} E_h.
$$

The redistribution of dividend income across countries generates country level differences between GDP_r and GNI_r, but at the global level these differences cancel out and the following relationship holds

$$GDP \equiv GNI \equiv \sum_{h\in\mathscr{H}} E_h. \tag{77}$$

2.6 Proof for Propositions in Section 4

2.6.1 Proof for Proposition 2

Using the following equations, I obtain a first-order approximation around the equilibrium for prices in sector $i \in \mathscr{N}$ and the bundle price for households of type $h \in \mathscr{H}$

$$p^{\ell}_{z_i} = \frac{\sum_{h\in\mathscr{H}} w_h\, \ell_{z_i h}}{A^{\ell}_i\, Q^{\ell}_i \left(\left\{A^{\ell}_{ih}\, \ell_{z_i h}\right\}_{h\in\mathscr{H}}\right)}, \tag{78}$$

$$p^{x}_{z_i} = \frac{\sum_{j\in\mathscr{N}} p_j\, x_{z_i j}}{A^{x}_i\, Q^{x}_i \left(\left\{A^{x}_{ij}\, x_{z_i j}\right\}_{j\in\mathscr{N}}\right)}, \tag{79}$$

$$p_{z_i} = \frac{\left(p^{\ell}_{z_i}\, L_{z_i} + p^{x}_{z_i}\, X_{z_i}\right)}{\mu_i\, A_i\, Q_i\left(L_{z_i},\, X_{z_i}\right)}, \tag{80}$$

$$p^{c}_h = \frac{\sum_{i\in\mathscr{N}} p_i\, C_{hi}}{Q^{c}_h\left(\left\{C_{hi}\right\}_{i\in\mathscr{N}}\right)}. \tag{81}$$

From equation (78)

$$\widehat{p}^{\ell}_{z_i} = \frac{A^{\ell}_i}{p^{\ell}_{z_i}}\overline{\frac{\partial\, p^{\ell}_{z_i}}{\partial\, A^{\ell}_i}}\widehat{A}^{\ell}_i + \sum_{h\in\mathscr{H}}\left(\frac{w_h}{p^{\ell}_{z_i}}\overline{\frac{\partial\, p^{\ell}_{z_i}}{\partial\, w_h}}\widehat{w}_h + \frac{A^{\ell}_{ih}}{p^{\ell}_{z_i}}\overline{\frac{\partial\, p^{\ell}_{z_i}}{\partial\, A^{\ell}_{ih}}}\widehat{A}^{\ell}_{ih} + \frac{\ell_{z_i h}}{p^{\ell}_{z_i}}\overline{\frac{\partial\, p^{\ell}_{z_i}}{\partial\, \ell_{z_i h}}}\widehat{\ell}_{z_i h}\right),$$

where $\frac{A^{\ell}_i}{p^{\ell}_{z_i}}\overline{\frac{\partial\, p^{\ell}_{z_i}}{\partial\, A^{\ell}_i}} = -1$, $\frac{w_h}{p^{\ell}_{z_i}}\overline{\frac{\partial\, p^{\ell}_{z_i}}{\partial\, w_h}} = \alpha_{z_i h}$, $\frac{A^{\ell}_{ih}}{p^{\ell}_{z_i}}\overline{\frac{\partial\, p^{\ell}_{z_i}}{\partial\, A^{\ell}_{ih}}} = -\alpha_{z_i h}$, $\frac{\ell_{z_i h}}{p^{\ell}_{z_i}}\overline{\frac{\partial\, p^{\ell}_{z_i}}{\partial\, \ell_{z_i h}}} = \alpha_{z_i h} - e\left(L_{z_i}, \ell_{z_i h}\right) = 0$ from equation (33), and $\widehat{x} = log\left(x/\overline{x}\right)$ stands for the log deviation around the equilibrium for variable x. As a consequence

$$\widehat{p}^{\ell}_{z_i} = -\widehat{A}^{\ell}_i + \sum_{h\in\mathscr{H}} \alpha_{z_i h}\left(\widehat{w}_h - \widehat{A}^{\ell}_{ih}\right). \tag{82}$$

Similarly, from equations (79), (80), and (81)

$$\widehat{p}^{x}_{z_i} = -\widehat{A}^{x}_i + \sum_{j\in\mathscr{N}} \omega_{z_i j}\left(\widehat{p}_j - \widehat{A}^{x}_{ij}\right), \tag{83}$$

$$\widehat{p}_{z_i} = \omega_{z_i}^{\ell}\,\widehat{p}_{z_i}^{\ell} + \omega_{z_i}^{x}\,\widehat{p}_{z_i}^{x} - \widehat{A}_i - \widehat{\mu}_i, \tag{84}$$

$$\widehat{p}_h^{c} = \sum_{q\in\mathscr{R}}\sum_{i\in\mathscr{N}_q} \beta_{hi}\,\widehat{p}_i. \tag{85}$$

From imposing symmetry in the decision of monopolistically competitive firms within the same sector, these equations are represented in matrix form by

$$\widehat{p}_{\ell} = \alpha\,\widehat{w} - \widehat{A}_{\ell} - \left(\alpha \circ \underline{\widehat{A}}_{\ell}\right)\mathbb{1}_H, \tag{86}$$

$$\widehat{p}_{x} = \mathscr{W}\,\widehat{p} - \widehat{A}_{x} - \left(\mathscr{W} \circ \underline{\widehat{A}}_{x}\right)\mathbb{1}_N, \tag{87}$$

$$\widehat{p} = diag\left(\omega_{\ell}\right)\widehat{p}_{\ell} + diag\left(\omega_{x}\right)\widehat{p}_{x} - \widehat{A} - \widehat{\mu}, \tag{88}$$

$$\widehat{p}_c = \beta\,\widehat{p}. \tag{89}$$

Introducing equations (86) and (87) in equation (88)

$$\widehat{p} = \widetilde{\Psi}_x\left(\widetilde{\Omega}_{\ell}\,\widehat{w} - \widehat{\mathscr{A}} - \widehat{\mu}\right) \tag{90}$$

and introducing equation (90) in equation (89)

$$\widehat{p}_c = \widetilde{\mathscr{C}}\,\widehat{w} - \widetilde{\mathscr{B}}\left(\widehat{\mathscr{A}} + \widehat{\mu}\right). \tag{91}$$

The matrices previously used are defined by

$$\alpha \equiv \begin{pmatrix} \alpha_{11} & \cdots & \alpha_{1H} \\ \vdots & \ddots & \vdots \\ \alpha_{N1} & \cdots & \alpha_{NH} \end{pmatrix}, \qquad \mathscr{W} \equiv \begin{pmatrix} \omega_{11} & \cdots & \omega_{1N} \\ \vdots & \ddots & \vdots \\ \omega_{N1} & \cdots & \omega_{NN} \end{pmatrix},$$

$$\widetilde{\Psi}_x \equiv \begin{pmatrix} \widetilde{\psi}_{11}^{x} & \cdots & \widetilde{\psi}_{1N}^{x} \\ \vdots & \ddots & \vdots \\ \widetilde{\psi}_{N1}^{x} & \cdots & \widetilde{\psi}_{NN}^{x} \end{pmatrix}, \qquad \widetilde{\mathscr{B}} \equiv \beta\,\widetilde{\Psi}_x \equiv \begin{pmatrix} \widetilde{\mathscr{B}}_{11} & \cdots & \widetilde{\mathscr{B}}_{1N} \\ \vdots & \ddots & \vdots \\ \widetilde{\mathscr{B}}_{H1} & \cdots & \widetilde{\mathscr{B}}_{HN} \end{pmatrix},$$

$\widehat{\mathscr{A}} \equiv \widehat{A} + diag\left(\omega_{\ell}\right)\widehat{A}_{\ell} + \left(\widetilde{\Omega}_{\ell} \circ \underline{\widehat{A}}_{\ell}\right)\mathbb{1}_H + diag\left(\omega_{x}\right)\widehat{A}_{x} + \left(\widetilde{\Omega}_{x} \circ \underline{\widehat{A}}_{x}\right)\mathbb{1}_N$, $\widehat{A} \equiv \left[\widehat{A}_1, \cdots, \widehat{A}_N\right]'$, $\widehat{A}_{\ell} \equiv \left[\widehat{A}_1^{\ell}, \cdots, \widehat{A}_N^{\ell}\right]'$, $\widehat{A}_{x} \equiv \left[\widehat{A}_1^{x}, \cdots, \widehat{A}_N^{x}\right]'$, $\underline{\widehat{A}}_{\ell} = \left[\underline{\widehat{A}}_1^{\ell}, \cdots, \underline{\widehat{A}}_N^{\ell}\right]'$, $\underline{\widehat{A}}_i^{\ell} = \left[\widehat{A}_{i1}^{\ell}, \cdots, \widehat{A}_{iH}^{\ell}\right]'$, $\underline{\widehat{A}}_x = \left[\underline{\widehat{A}}_1^{x}, \cdots, \underline{\widehat{A}}_n^{x}\right]'$, $\underline{\widehat{A}}_i^{x} = \left[\widehat{A}_{i1}^{x}, \cdots, \widehat{A}_{iN}^{x}\right]'$, $\widehat{p} \equiv \left[\widehat{p}_1, \cdots, \widehat{p}_N\right]'$, $\widehat{p}_{\ell} \equiv \left[\widehat{p}_1^{\ell}, \cdots, \widehat{p}_N^{\ell}\right]'$, $\widehat{p}_x \equiv \left[\widehat{p}_1^{x}, \cdots, \widehat{p}_N^{x}\right]'$, and $\widehat{\mu} \equiv \left[\widehat{\mu}_1, \cdots, \widehat{\mu}_N\right]'$.

2.6.2 Proof for Theorem 1

From equations (69) and (70)

$$\Lambda_h \widehat{\Gamma}_h = \mathbb{1}_N' \, diag\left(\Omega_\ell \, o_H\left(h\right)\right) \mathscr{B}' \left(\begin{pmatrix} \chi_1 \, \widehat{\chi}_1 \\ \vdots \\ \chi_H \, \widehat{\chi}_H \end{pmatrix} - \chi \, \widehat{\chi}_h \right) + \sum_{i \in \mathscr{N}} \Omega_{ih}^{\ell} \, \lambda_i \, \left(\widehat{\omega}_i^{\ell} + \widehat{\alpha}_{ih} \right)$$
$$+ \mathbb{1}_N' \, diag\left(\Omega_\ell \, o_H\left(h\right)\right) \Psi_x' \begin{pmatrix} \sum_{b \in \mathscr{H}} \beta_{b1} \, \chi_b \, \widehat{\beta}_{b1} \\ \vdots \\ \sum_{b \in \mathscr{H}} \beta_{bN} \, \chi_b \, \widehat{\beta}_{bN} \end{pmatrix}$$
$$+ \mathbb{1}_N' \, diag\left(\widetilde{\Omega}_\ell \, o_H\left(h\right)\right) \frac{d\left(diag\left(\mu\right)^{-1} - \widetilde{\Omega}_x'\right)^{-1}}{d \log \widetilde{\Omega}_x} \beta' \chi$$
$$+ \mathbb{1}_N' \, diag\left(\widetilde{\Omega}_\ell \, o_H\left(h\right)\right) \frac{d\left(diag\left(\mu\right)^{-1} - \widetilde{\Omega}_x'\right)^{-1}}{d \log \mu} \beta' \chi$$

Using equations (59), (63), (66), and (65), and the fact that for any invertible matrix A, $\frac{dA^{-1}}{dx} = -A^{-1}\frac{dA}{dx}A^{-1}$, the previous equation becomes

$$\widehat{\Gamma}_h = \sum_{b \in \mathscr{H}} \mathscr{C}_{bh} \frac{\chi_b}{\Lambda_h} \widehat{\chi}_b - \widehat{\chi}_h + \Lambda_h^{-1} \sum_{i \in \mathscr{N}} \Omega_{ih}^{\ell} \sum_{j \in \mathscr{N}} \Psi_{ji}^{x} \sum_{b \in \mathscr{H}} \beta_{bj} \, \chi_b \, \widehat{\beta}_{bj} + \sum_{i \in \mathscr{N}} \Omega_{ih}^{\ell} \, \lambda_i \, \left(\widehat{\omega}_i^{\ell} + \widehat{\alpha}_{ih} \right)$$
$$- \Lambda_h^{-1} \, \mathbb{1}_N' diag\left(\Omega_\ell \, o_H\left(h\right)\right) \Psi_x' \frac{d\left(diag\left(\mu\right)^{-1} - \widetilde{\Omega}_x'\right)}{d \log \widetilde{\Omega}_x} diag\left(\mu\right) \, \lambda$$
$$- \Lambda_h^{-1} \, \mathbb{1}_N' diag\left(\Omega_\ell \, o_H\left(h\right)\right) \Psi_x' \frac{d\left(diag\left(\mu\right)^{-1} - \widetilde{\Omega}_x'\right)}{d \log \mu} diag\left(\mu\right) \, \lambda.$$

$$\begin{aligned} \widehat{\Gamma}_h &= \sum_{b \in \mathscr{H}} \mathscr{C}_{bh} \frac{\chi_b}{\Lambda_h} \widehat{\chi}_b - \widehat{\chi}_h + \sum_{b \in \mathscr{H}} \mathscr{C}_{bh} \frac{\chi_b}{\Lambda_h} \widehat{\mathscr{C}}_{bh} \\ &= \sum_{b \in \mathscr{H}} \mathscr{C}_{bh} \frac{\chi_b}{\Lambda_h} \widehat{\chi}_b - \widehat{\chi}_h + \frac{1}{\Lambda_h} o_H\left(h\right)' \, \Psi_\ell' \, diag\left(\widehat{\mu}\right) \, \lambda \\ &+ \frac{1}{\Lambda_h} \left(\sum_{i \in \mathscr{N}} \Omega_{ih}^{\ell} \, \lambda_i \left(\widehat{\omega}_i^{\ell} + \widehat{\alpha}_{ih} \right) + \sum_{j \in \mathscr{N}} \psi_{jh}^{\ell} \left(\sum_{b \in \mathscr{H}} \beta_{bj} \, \chi_b \, \widehat{\beta}_{bj} + \sum_{i \in \mathscr{N}} \Omega_{ij}^{x} \, \lambda_i \, \left(\widehat{\omega}_i^{x} + \widehat{\omega}_{ij} \right) \right) \right). \end{aligned} \tag{92}$$

Now, using equation (70)

$$\begin{aligned} d\,\Lambda_h &= \sum_{b \in \mathscr{H}} \mathscr{C}_{bh} \, d\,\chi_b + \sum_{b \in \mathscr{H}} \chi_b \, d\,\mathscr{C}_{bh} \\ &= \sum_{b \in \mathscr{H}} \mathscr{C}_{bh} \, d\,\chi_b + \sum_{i \in \mathscr{N}} \psi_{ih}^{\ell} \, \lambda_i \, d \log \mu_i + \sum_{i \in \mathscr{N}} \lambda_i \, \mu_i \, d\,\widetilde{\Omega}_{ih}^{\ell} + \sum_{j \in \mathscr{N}} \psi_{jh}^{\ell} \left(\sum_{b \in \mathscr{H}} \chi_b \, d\,\beta_{bj} + \sum_{i \in \mathscr{N}} \lambda_i \, \mu_i \, d\,\widetilde{\Omega}_{ij}^{x} \right). \end{aligned} \tag{93}$$

2.6.3 Proof for idiosyncratic Positional Terms of Trade

The first order approximation for equation (38) for household $h \in \mathscr{H}$ is given by

$$\widehat{E}_h = \Gamma_h \left(\widehat{w}_h + \widehat{L}_h\right) + \frac{\Pi_h}{E_H}\widehat{\Pi}_h + \frac{T_h}{E_h}\,\widehat{T}_h. \tag{94}$$

The first order approximation for dividend income in equations (39) and (71) is given by

$$\Pi_h\,\widehat{\Pi}_h = \sum_{i\in\mathscr{N}} \kappa_{ih} \int S_{z_i} \left((1-\mu_i)\left(\widehat{\kappa}_{ih} + \widehat{S}_{z_i}\right) dz_i - \mu_i\,\widehat{\mu}_i\right) dz_i. \tag{95}$$

Introducing equation (95) in equation (94)

$$E_h\,\widehat{E}_h = J_h \left(\widehat{w}_h + \widehat{L}_h\right) + \sum_{i\in\mathscr{N}} \kappa_{ih} \int S_{z_i} \left((1-\mu_i)\left(\widehat{\kappa}_{ih} + \widehat{S}_{z_i}\right) - \mu_i\,\widehat{\mu}_i\right) dz_i + T_h\,\widehat{T}_h. \tag{96}$$

From equations (91) and (96), and imposing symmetry in the decision of monopolistically competitive firms within the same sector

$$\begin{aligned}\widehat{C}_h &= \widehat{E}_h - \widehat{p}^c_h \\ &= \Gamma_h \left(\widehat{w}_h + \widehat{L}_h\right) + \sum_{i\in\mathscr{N}} \kappa_{ih}\,\frac{\lambda_i}{\chi_h} \left((1-\mu_i)\left(\widehat{\kappa}_{ih} + \widehat{S}_i\right) - \mu_i\,\widehat{\mu}_i\right) + \frac{\mathbb{T}_h}{\chi_h}\,\widehat{T}_h - \widetilde{\mathscr{C}}'_h\,\widehat{w} + \widetilde{\mathscr{B}}'_h\left(\widehat{\mathscr{A}} + \widehat{\mu}\right).\end{aligned}$$

where $\widetilde{\mathscr{B}}_h = \left[\widetilde{\mathscr{B}}_{h1}, \cdots, \widetilde{\mathscr{B}}_{hN}\right]'$, and $\widetilde{\mathscr{C}}_h = \left[\widetilde{\mathscr{C}}_{h1}, \cdots, \widetilde{\mathscr{C}}_{hH}\right]'$. Then

$$\widehat{C}_h = \widetilde{\mathscr{B}}'_h\left(\widehat{\mathscr{A}} + \widehat{\mu}\right) + \Gamma_h\,\widehat{J}_h - \widetilde{\mathscr{C}}'_h\,\widehat{J} + \sum_{i\in\mathscr{N}} \kappa_{ih}\,\frac{\lambda_i}{\chi_h} \left((1-\mu_i)\left(\widehat{\kappa}_{ih} + \widehat{S}_i\right) - \mu_i\,\widehat{\mu}_i\right) + \frac{\mathbb{T}_h}{\chi_h}\,\widehat{T}_h + \widetilde{\mathscr{C}}'_h\,\widehat{L}.$$

Therefore

$$\frac{C_h}{\overline{C}_h} = \overline{\eta}_h\,\mathscr{D}_h(\mathscr{A})\,\mathscr{D}_h(\mu)\,\mathscr{D}_h(J)\,\mathscr{D}_h(\Pi)\,\mathscr{D}_h(T)\;\;f_h\left(\{L_b\}_{b\in\mathscr{H}}\right) \tag{97}$$

where $f_h\left(\{L_b\}_{b\in\mathscr{H}}\right)$ is a CRS function such that $\frac{d\,log\,f_h\left(\{L_b\}_{b\in\mathscr{H}}\right)}{d\,log\,L_b} = \widetilde{\mathscr{C}}_{hb}$, and

$$\mathscr{D}_h(\mathscr{A}) = exp\left\{\widetilde{\mathscr{B}}'_h\,\widehat{\mathscr{A}}\right\}, \qquad \mathscr{D}_h(\mu) = exp\left\{\widetilde{\mathscr{B}}'_h\,\widehat{\mu}\right\}, \qquad \mathscr{D}_h(T) = exp\left\{\frac{\mathbb{T}_h}{\chi_h}\,\widehat{T}_h\right\}$$

$$\mathscr{D}_h(\Pi) = exp\left\{\sum_{q\in\mathscr{R}}\sum_{i\in\mathscr{N}_q} \kappa_{ih}\,\frac{\lambda_i}{\chi_h} \left((1-\mu_i)\left(\widehat{\kappa}_{ih} + \widehat{S}_i\right) - \mu_i\,\widehat{\mu}_i\right)\right\},$$

$$\mathscr{D}_h\left(J\right)=exp\left\{\Gamma_h\,\widehat{J}_h-\widetilde{\mathscr{C}}_h'\,\widehat{J}\right\},\tag{98}$$

and $\overline{\eta}_h$ stands for a constant.

As a consequence

$$\begin{aligned}C_h&=\eta_h\,\mathscr{D}_h\left(\mathscr{A}\right)\,\mathscr{D}_h\left(\mu\right)\,\mathscr{D}_h\left(J\right)\,\mathscr{D}_h\left(\Pi\right)\,\mathscr{D}_h\left(T\right)\,\mathscr{D}_h\left(\tau_\pi\right)\;f_h\left(\{L_b\}_{b\in\mathscr{H}}\right)\\&=PTT_h\,f_h\left(\{L_b\}_{b\in\mathscr{H}}\right)\end{aligned}\tag{99}$$

$$PTT_h=\eta_h\,\mathscr{D}_h\left(\mathscr{A}\right)\,\mathscr{D}_h\left(\mu\right)\,\mathscr{D}_h\left(J\right)\,\mathscr{D}_h\left(\Pi\right)\,\mathscr{D}_h\left(T\right)$$

with $\eta_h=\overline{\eta}_h\,\overline{C}_h$.

Add and substract $\widehat{GDP}$ to express equation (99) in terms of Domar weights and labor income shares

$$\begin{aligned}\widehat{C}_h&=\widetilde{\mathscr{B}}_h'\left(\widehat{\mathscr{A}}+\widehat{\mu}\right)+\Gamma_h\,\widehat{\Lambda}_h-\widetilde{\mathscr{C}}_h'\,\widehat{\Lambda}\\&\quad+\sum_{i\in\mathscr{N}}\kappa_{ih}\,\frac{\lambda_i}{\chi_h}\left((1-\mu_i)\left(\widehat{\kappa}_{ih}+\widehat{\lambda}_i\right)-\mu_i\,\widehat{\mu}_i\right)+\frac{\mathbb{T}_h}{\chi_h}\,\widehat{\mathbb{T}}_h+\widetilde{\mathscr{C}}_h'\,\widehat{L}\\&\quad+\underbrace{\left(\Gamma_h+\sum_{i\in\mathscr{N}}\kappa_{ih}\,\frac{\lambda_i}{\chi_h}\,(1-\mu_i)+\frac{\mathbb{T}_h}{\chi_h}-\sum_{b\in\mathscr{H}}\widetilde{\mathscr{C}}_{hb}\right)}_{=0}\widehat{GDP}\\&=\widetilde{\mathscr{B}}_h'\left(\widehat{\mathscr{A}}+\widehat{\mu}\right)+\Gamma_h\,\widehat{\Lambda}_h-\widetilde{\mathscr{C}}_h'\,\widehat{\Lambda}\\&\quad+\sum_{i\in\mathscr{N}}\kappa_{ih}\,\frac{\lambda_i}{\chi_h}\left((1-\mu_i)\left(\widehat{\kappa}_{ih}+\widehat{\lambda}_i\right)-\mu_i\,\widehat{\mu}_i\right)+\frac{\mathbb{T}_h}{\chi_h}\,\widehat{\mathbb{T}}_h+\widetilde{\mathscr{C}}_h'\,\widehat{L}\end{aligned}$$

where the last equality is given by equations (66) and (74).

The $N+2$ vector $\mathscr{R}_h$ captures the revenue distribution for household h

$$\begin{aligned}\mathscr{R}_h'&=\begin{bmatrix}\Gamma_h & \frac{\mathbb{T}_h}{\chi_h} & \frac{1}{\chi_h}\lambda' diag\left(\Omega_\pi\, o_H\left(h\right)\right)\end{bmatrix}\\&=\frac{1}{\chi_h}\begin{bmatrix}\Lambda_h & \mathbb{T}_h & \kappa_{1h}\left(1-\mu_1\right)\lambda_1 & \cdots & \kappa_{Nh}\left(1-\mu_N\right)\lambda_N\end{bmatrix}.\end{aligned}$$

The first element captures the share of labor income in household h's income, the second element represents the share of transfers in household hs income, and the last N elements capture the share of profits by each sector on household h's income. As the elements of this vector add up to one, its first-order approximation is given by

$$\widehat{\chi}_h=\Gamma_h\,\widehat{\Lambda}_h+\sum_{i\in\mathscr{N}}\kappa_{ih}\,\frac{\lambda_i}{\chi_h}\left((1-\mu_i)\left(\widehat{\kappa}_{ih}+\widehat{\lambda}_i\right)-\mu_i\,\widehat{\mu}_i\right)+\frac{\mathbb{T}_h}{\chi_h}\widehat{\mathbb{T}}_h.\tag{100}$$

This implies that

$$\widehat{C}_h = \widetilde{\mathscr{B}}_h' \left(\widehat{\mathcal{A}} + \widehat{\mu}\right) + \widehat{\chi}_h - \widetilde{\mathscr{C}}_h' \,\widehat{\Lambda} + \widetilde{\mathscr{C}}_h' \,\widehat{L}. \tag{101}$$

Now, using equations (70) and (93), and the definitions $\delta_{b|h} = \widetilde{\mathscr{C}}_{hb}/\widetilde{\Lambda}_b$, $M_{q|h} = \sum_{b\in\mathscr{H}} \mathscr{C}_{qb}\,\delta_{b|h}$, and $F_{i|h} = \sum_{q\in\mathscr{H}} \psi^{\ell}_{iq}\,\delta_{q|h}$

$$\begin{aligned}
\sum_{b\in\mathscr{H}} \mathscr{C}_{hb}\,\widehat{\Lambda}_b &= \sum_{b\in\mathscr{H}} \delta_{b|h}\, d\,\widehat{\Lambda}_b \\
&= \sum_{b\in\mathscr{H}} M_{b|h}\, d\,\chi_b + \sum_{m\in\mathscr{H}} \chi_m \sum_{b\in\mathscr{H}} \delta_{b|h}\, d\,\mathscr{C}_{mb} \\
&= \sum_{b\in\mathscr{H}} M_{b|h}\, d\,\chi_b + \sum_{i\in\mathscr{N}} \lambda_i\, F_{i|h}\, d\log\mu_i + \sum_{i\in\mathscr{N}} \mu_i\,\lambda_i \sum_{b\in\mathscr{H}} \delta_{b|h}\, d\,\widetilde{\Omega}^{\ell}_{ib} \\
&+ \sum_{b\in\mathscr{H}} \chi_b \sum_{i\in\mathscr{N}} F_{i|h}\, d\,\beta_{bi} + \sum_{i\in\mathscr{N}} \mu_i\,\lambda_i \sum_{j\in\mathscr{N}} F_{j|h}\, d\,\widetilde{\Omega}^{x}_{ij}.
\end{aligned}$$

Hence

$$\begin{aligned}
\widehat{PTT}_h &= \widetilde{\mathscr{B}}_h' \left(\widehat{\mathcal{A}} + \widehat{\mu}\right) + \widehat{\chi}_h - \sum_{b\in\mathscr{H}} M_{b|h}\, d\,\chi_b - \sum_{i\in\mathscr{N}} \lambda_i\, F_{i|h}\, d\log\mu_i \\
&- \sum_{i\in\mathscr{N}} \mu_i\,\lambda_i \sum_{b\in\mathscr{H}} \delta_{b|h}\, d\,\widetilde{\Omega}^{\ell}_{ib} - \sum_{b\in\mathscr{H}} \chi_b \sum_{i\in\mathscr{N}} F_{i|h}\, d\,\beta_{bi} - \sum_{i\in\mathscr{N}} \mu_i\,\lambda_i \sum_{j\in\mathscr{N}} F_{j|h}\, d\,\widetilde{\Omega}^{x}_{ij}.
\end{aligned} \tag{102}$$

This represents an extension of the positional terms of trade introduced by Rojas-Bernal (2023) to an open economy setting with sectoral distortion.

2.6.4 First-order variation for GNI_r

$GNI_r \equiv \sum_{h\in\mathscr{H}_r} E_h$. This implies that

$$\Phi^{GNI}_r = \sum_{h\in\mathscr{H}_r} \chi_h = \sum_{h\in\mathscr{H}_r} \left(\Lambda_h + \sum_{i\in\mathscr{N}} \kappa_{ih}\,(1-\mu_i)\,\lambda_i + \mathbb{T}_h\right),$$

where $\Phi^{GNI}_r = GNI_r/GDP$.

Now, from equations (49) we know that

$$\sum_{h\in\mathscr{H}_r} \mathbb{T}_h = 0$$

Hence

$$\Phi_r^{GNI} = \sum_{h \in \mathscr{H}_r} \left(\Lambda_h + \sum_{i \in \mathscr{N}} \kappa_{ih} \left(1 - \mu_i\right) \lambda_i \right).$$

The first-order approximation for the last equation tells us that

$$\Phi_r^{GNI} \, \widehat{\Phi}_r^{GNI} = \sum_{h \in \mathscr{H}_r} \left(\Lambda_h \, \widehat{\Lambda}_h + \sum_{i \in \mathscr{N}} \kappa_{ih} \lambda_i \left(\left(1 - \mu_i\right) \left(\widehat{\kappa}_{ih} + \widehat{\lambda}_i \right) - \mu_i \, \widehat{\mu}_i \right) \right). \tag{103}$$

2.6.5 Proof for Corollary 2

Starting from equation (75), I arrive to the first-order approximation for nominal GDP in country r

$$GDP_r \widehat{GDP}_r = \sum_{i \in \mathscr{N}_r} S_i \left(\widehat{S}_i - \sum_{j \in \mathscr{N}} \mu_i \, \omega_i^x \, \omega_{ij} \left(\widehat{\mu}_i + \widehat{\omega}_i^x + \widehat{\omega}_{ij} + \widehat{S}_i \right) \right).$$

From equations (30) and (87)

$$\Phi_r \, \widehat{GDP}_r = \sum_{i \in \mathscr{N}_r} \lambda_i \left(\widehat{S}_i - \sum_{j \in \mathscr{N}} \Omega_{ij}^x \left(\widehat{p}_j + \widehat{x}_{ij} \right) \right), \tag{104}$$

where $\Phi_r = \frac{GDP_r}{GDP}$.

From here, I define the first-order approximation of the GDP deflator from country r as

$$\widehat{p}_{Y_r} = \sum_{i \in \mathscr{N}_r} \frac{\lambda_i}{\Phi_r} \left(\widehat{p}_i - \sum_{j \in \mathscr{N}} \Omega_{ij}^x \, \widehat{p}_j \right). \tag{105}$$

Equation (105) implies using equation (34) that the first-order approximation of the real GDP from country r is given by

$$\Phi_r \, \widehat{Y}_r = \sum_{i \in \mathscr{N}_r} \lambda_i \left(\widehat{y}_i - \sum_{j \in \mathscr{N}} \Omega_{ij}^x \, \widehat{x}_{ij} \right). \tag{106}$$

Using the first-order approximation for the goods market clearing condition in equation (52), equation (106) is represented by

$$\Phi_r\,\widehat{Y}_r = \sum_{i\in\mathscr{N}_r}\left(\sum_{h\in\mathscr{H}}\beta_{hi}\,\chi_h\,\widehat{C}_{hi} + \sum_{j\in\mathscr{N}}\left(\Omega^x_{ji}\,\lambda_j\,\widehat{x}_{ji} - \Omega^x_{ij}\,\lambda_i\,\widehat{x}_{ij}\right)\right),$$

introducing the first-order approximation of equations (30), (34), (46), (57), (90), and (96)

$$\begin{aligned}\Phi_r\,\widehat{Y}_r = {} & \sum_{h\in\mathscr{H}}\chi_h\sum_{i\in\mathscr{N}_r}\beta_{hi}\,\widehat{\beta}_{hi} + \sum_{h\in\mathscr{H}}\chi_h\left(\sum_{i\in\mathscr{N}_r}\beta_{hi}\right)\widehat{E}_h + \sum_{j\in\mathscr{N}}\mu_j\,\omega^x_j\,\lambda_j\sum_{i\in\mathscr{N}_r}\omega_{ji}\,\widehat{\omega}_{ji} - \sum_{i\in\mathscr{N}_r}\mu_i\,\omega^x_i\,\lambda_i\sum_{j\in\mathscr{N}}\omega_{ij}\,\widehat{\omega}_{ij}\\ & + \sum_{i\in\mathscr{N}_r}\sum_{j\in\mathscr{N}}\Omega^x_{ji}\,\lambda_j\left(\widehat{\omega}^x_j + \widehat{\mu}_j\right) - \sum_{i\in\mathscr{N}_r}\sum_{j\in\mathscr{N}}\Omega^x_{ij}\,\lambda_i\left(\widehat{\omega}^x_i + \widehat{\mu}_i\right) + \sum_{i\in\mathscr{N}_r}\sum_{j\in\mathscr{N}}\Omega^x_{ji}\,\lambda_j\,\widehat{S}_j - \sum_{i\in\mathscr{N}_r}\lambda_i\left(\sum_{j\in\mathscr{N}}\Omega^x_{ij}\right)\widehat{S}_i\\ & + \sum_{i\in\mathscr{N}_r}\mu_i\,\omega^x_i\,\lambda_i\sum_{q\in\mathscr{R}}\sum_{j\in\mathscr{N}_q}\omega_{ij}\,\widehat{p}_j - \sum_{i\in\mathscr{N}_r}\underbrace{\left(\sum_{h\in\mathscr{H}}\beta_{hi}\,\chi_h + \sum_{j\in\mathscr{N}}\Omega^x_{ji}\,\lambda_j\right)}_{=\lambda_i}\widehat{p}_i.\end{aligned}$$

$$\begin{aligned}\Phi_r\,\widehat{Y}_r = {} & \sum_{h\in\mathscr{H}}\chi_h\sum_{i\in\mathscr{N}_r}\beta_{hi}\,\widehat{\beta}_{hi} + \sum_{h\in\mathscr{H}}\left(\sum_{j\in\mathscr{N}_r}\beta_{hj}\right)\left(\Lambda_h\left(\widehat{w}_h + \widehat{L}_h\right) + \mathbb{T}_h\,\widehat{T}_h\right)\\ & + \sum_{h\in\mathscr{H}}\left(\sum_{j\in\mathscr{N}_r}\beta_{hj}\right)\sum_{i\in\mathscr{N}}\kappa_{ih}\,\lambda_i\left((1-\mu_i)\left(\widehat{\kappa}_{ih} + \widehat{S}_i\right) - \mu_i\,\widehat{\mu}_i\right)\\ & + \sum_{j\in\mathscr{N}}\mu_j\,\omega^x_j\,\lambda_j\sum_{i\in\mathscr{N}_r}\omega_{ji}\,\widehat{\omega}_{ji} - \sum_{i\in\mathscr{N}_r}\mu_i\,\omega^x_i\,\lambda_i\sum_{j\in\mathscr{N}}\omega_{ij}\,\widehat{\omega}_{ij}\\ & + \sum_{j\in\mathscr{N}}\mu_j\,\omega^x_j\,\lambda_j\left(\sum_{i\in\mathscr{N}_r}\omega_{ji}\right)\left(\widehat{\omega}^x_j + \widehat{\mu}_j\right) - \sum_{i\in\mathscr{N}_r}\mu_i\,\omega^x_i\,\lambda_i\left(\sum_{j\in\mathscr{N}}\omega_{ij}\right)\left(\widehat{\omega}^x_i + \widehat{\mu}_i\right)\\ & + \sum_{i\in\mathscr{N}_r}\sum_{j\in\mathscr{N}}\Omega^x_{ji}\,\lambda_j\,\widehat{S}_j - \sum_{i\in\mathscr{N}_r}\lambda_i\left(\sum_{j\in\mathscr{N}}\Omega^x_{ij}\right)\widehat{S}_i\\ & + \sum_{j\in\mathscr{N}}\left(\sum_{i\in\mathscr{N}_r}\lambda_i\left(\widetilde{\psi}^x_{ij} - \sum_{m\in\mathscr{N}}\Omega^x_{im}\,\widetilde{\psi}^x_{mj}\right)\right)\left(\widehat{\mathscr{A}}_j + \widehat{\mu}_j\right)\\ & - \sum_{h\in\mathscr{H}}\left(\sum_{i\in\mathscr{N}_r}\lambda_i\left(\widetilde{\psi}^\ell_{ih} - \sum_{m\in\mathscr{N}}\Omega^x_{im}\,\widetilde{\psi}^\ell_{mh}\right)\right)\left(\widehat{w}_h + \widehat{L}_h\right)\\ & + \sum_{h\in\mathscr{H}}\left(\sum_{i\in\mathscr{N}_r}\lambda_i\left(\widetilde{\psi}^\ell_{ih} - \sum_{m\in\mathscr{N}}\Omega^x_{im}\,\widetilde{\psi}^\ell_{mh}\right)\right)\widehat{L}_h\end{aligned}$$

Before continuing with this proof, let me define the following variables

- $\widetilde{\Psi}_x = \sum_{q=0}^{\infty}\widetilde{\Omega}^q_x = I_N + \widetilde{\Omega}_x\widetilde{\Psi}_x$ implies that $\widetilde{\Psi}_x - \Omega_x\widetilde{\Psi}_x = I_N + \left(\widetilde{\Omega}_x - \Omega_x\right)\widetilde{\Psi}_x$. On the one hand, in the absence of distortions $\widetilde{\psi}^x_{ij} - \sum_{m\in\mathscr{N}}\Omega^x_{im}\widetilde{\psi}^x_{mj} = \mathbb{1}\{i=j\}$ captures direct exposure to a shocks. This implies that in the absence of distortions a shock in sector i

with magnitude $\widetilde{\psi}^x_{ij} - \sum_{m\in\mathcal{N}} \Omega^x_{im}\widetilde{\psi}^x_{mj} = \mathbb{1}\{i=j\}$ has a direct effect on the real GDP of country r that is to a first-order proportional to $\mathbb{1}\{i \in \mathcal{N}_r\}$. As a consequence, without distortions, there are no spillover to other countries.

On the other hand, in an economy with distortions, $\widetilde{\psi}^x_{ij} - \sum_{m\in\mathcal{N}} \Omega^x_{im}\widetilde{\psi}^x_{mj} = \mathbb{1}\{i=j\} + \sum_{m\in\mathcal{N}} \left(\widetilde{\Omega}^x_{im} - \Omega^x_{im}\right)\widetilde{\psi}^x_{mj}$ captures for sector i both the direct exposure and higher-round network adjusted effects from shocks in sector j, but the effects from the latter channel that come from firm m are weighted by the cost-to-revenue margin $\widetilde{\Omega}^x_{im} - \Omega^x_{im}$. In other words, $\widetilde{\psi}^x_{ij\,(\widetilde{\Omega}-\Omega)} = \sum_{m\in\mathcal{N}} \left(\widetilde{\Omega}^x_{im} - \Omega^x_{im}\right)\widetilde{\psi}^x_{mj}$ represents the network adjusted downstream exposure of sector i to sector j weighted by the difference between the direct downstream cost exposure and the direct upstream revenue exposure in each one of the paths through which intermediate input costs from firm j influence directly or indirectly the costs for firm i. This implies that under distortions, a shock in sector j with magnitude $\widetilde{\psi}^x_{ij} - \sum_{m\in\mathcal{N}} \Omega^x_{im}\widetilde{\psi}^x_{mj}$ has a direct effect on the real GDP of country r that is to a first-order proportional to $\mathbb{1}\{i=j\} + \widetilde{\psi}^x_{ij\,(\widetilde{\Omega}-\Omega)}$. As a consequence, under distortions, favorable shocks directly increase the real GDP of a country if the sector that receives the shock is a domestic producer, or if there are distorted domestic firms with some degree of downstream exposure to the sector that receives the shock. The latter channel captures the additional value-added that is captured by domestic firms with downstream exposure.

For this reason,

$$\ddot{\lambda}^r_j = \mathbb{1}\{j \in \mathcal{N}_r\}\frac{\lambda_j}{\Phi_r} + \sum_{i\in\mathcal{N}_r}\frac{\lambda_i}{\Phi_r}\widetilde{\psi}^x_{ij\,(\widetilde{\Omega}-\Omega)} = \mathbb{1}\{j \in \mathcal{N}_r\}\frac{\lambda_j}{\Phi_r} + \sum_{i\in\mathcal{N}_r}\frac{\lambda_i}{\Phi_r}\sum_{m\in\mathcal{N}}(1-\mu_i)\,\widetilde{\Omega}^x_{im}\widetilde{\psi}^x_{mj}$$

represents the share of value added in country r than can be traced-back to the production from sector j. Value added can be extracted in two ways. First, by producing and selling the good. Second, by importing intermediate goods, using them to produce domestic, and charging a surplus that is reflected in profits or taxes.

Notice that in the absence of intermediate inputs $\sum_{i\in\mathcal{N}_r}\ddot{\lambda}^r_i = 1$. In the absence of distortions and with intermediate inputs $\sum_{i\in\mathcal{N}_r}\ddot{\lambda}^r_i = \sum_{i\in\mathcal{N}_r}\frac{\lambda_i}{\Phi_r} \geq 1$. In general $\sum_{i\in\mathcal{N}_r}\ddot{\lambda}^r_i \geq 1$. For the global economy, these weights are given by $\ddot{\lambda}^G_j = \lambda_j + \sum_{i\in\mathcal{N}}\lambda_i\sum_{m\in\mathcal{N}}\left(\widetilde{\Omega}^x_{im} - \Omega^x_{im}\right)\widetilde{\psi}^x_{mj}$. In vector form are represented by $\ddot{\lambda}^G = \left(I_N + \widetilde{\Psi}'_x\left(\widetilde{\Omega}_x - \Omega_x\right)'\right)\lambda = \widetilde{\Psi}'_x\left(I_N - \Omega'_x\right)\lambda$.

We know that

$$\lambda = \Psi'_x\,\beta\,\chi = \Psi'_x\left(I_N - \widetilde{\Omega}'_x\right)\widetilde{\Psi}'_x\,\beta\,\chi = \Psi'_x\left(I_N - \widetilde{\Omega}'_x\right)\widetilde{\lambda}, \tag{107}$$

which implies that $\ddot{\lambda}^G = \widetilde{\lambda}$.

- Similarly $\widetilde{\Psi}_\ell - \Omega_x\widetilde{\Psi}_\ell = \widetilde{\Omega}_\ell + \left(\widetilde{\Omega}_x - \Omega_x\right)\widetilde{\Psi}_\ell$. On the one hand, in the absence of distortions $\widetilde{\psi}^\ell_{ih} - \sum_{j\in\mathcal{N}}\Omega^x_{ij}\widetilde{\psi}^\ell_{jh} = \widetilde{\Omega}^\ell_{ih}$ captures the direct exposure of firm i to labor costs from worker

h.

On the other hand, in an economy with distortions, $\widetilde{\Omega}^{\ell}_{ih} + \sum_{j\in\mathcal{N}} \left(\widetilde{\Omega}^{x}_{ij} - \Omega^{x}_{ij}\right) \widetilde{\psi}^{\ell}_{jh}$ captures for sector i both the direct exposure and higher-round network effects from labor costs from worker h. In other words, $\widetilde{\psi}^{\ell}_{ih\,(\widetilde{\Omega}-\Omega)} = \sum_{j\in\mathcal{N}} \left(\widetilde{\Omega}^{x}_{ij} - \Omega^{x}_{ij}\right) \widetilde{\psi}^{\ell}_{jh}$ represents the network adjusted downstream exposure of sector i to household h's factoral cost weighted by the difference between the direct downstream cost exposure and the direct upstream revenue exposure in each of the paths through which labor costs from worker h influence directly or indirectly the costs for firm i.

For this reason, $\ddot{\Lambda}^{r}_{h} = \sum_{i\in\mathcal{N}_r} \frac{\lambda_i}{\Phi_r} \left(\widetilde{\Omega}^{\ell}_{ih} + \widetilde{\psi}^{\ell}_{ih\,(\widetilde{\Omega}-\Omega)}\right)$ represents the share of value added in country r that can be traced back to the labor supply from workers of type h. Notice that this characterizes a distribution because

$$\sum_{h\in\mathcal{H}} \ddot{\Lambda}^{r}_{h} = \sum_{i\in\mathcal{N}_r} \frac{\lambda_i}{\Phi_r} \underbrace{\sum_{h\in\mathcal{H}} \widetilde{\Omega}^{\ell}_{ih}}_{=\omega^{\ell}_{i}} + \sum_{i\in\mathcal{N}_r} \frac{\lambda_i}{\Phi_r} \sum_{j\in\mathcal{N}} \left(\widetilde{\Omega}^{x}_{ij} - \Omega^{x}_{ij}\right) \underbrace{\sum_{h\in\mathcal{H}} \widetilde{\psi}^{\ell}_{jh}}_{=1}$$

$$= \sum_{i\in\mathcal{N}_r} \omega^{\ell}_{i} \frac{\lambda_i}{\Phi_r} + \sum_{i\in\mathcal{N}_r} \frac{\lambda_i}{\Phi_r} \underbrace{\sum_{j\in\mathcal{N}} \widetilde{\Omega}^{x}_{ij}}_{=\omega^{\ell}_{i}} - \sum_{i\in\mathcal{N}_r} \frac{\lambda_i}{\Phi_r} \sum_{j\in\mathcal{N}} \Omega^{x}_{ij} = \sum_{i\in\mathcal{N}_r} \left(1 - \sum_{j\in\mathcal{N}} \Omega^{x}_{ij}\right) \frac{\lambda_i}{\Phi_r} = 1,$$

where the last equality is given by equation (75).

Without distortions, $\widetilde{\Omega}^{\ell}_{ih} = \Omega^{\ell}_{ih}$

$$\ddot{\Lambda}^{r}_{h} = \frac{1}{\Phi_r} \sum_{i\in\mathcal{N}_r} \Omega^{\ell}_{ih}\, \lambda_i,$$

and with country-specific factors

$$\ddot{\Lambda}^{r}_{h} = \frac{\Lambda_h}{\Phi_r}.$$

For the global economy, these weights are given by $\ddot{\Lambda}^{G}_{h} = \sum_{i\in\mathcal{H}} \lambda_i \left(\widetilde{\Omega}^{\ell}_{ih} + \sum_{j\in\mathcal{N}} \left(\widetilde{\Omega}^{x}_{ij} - \Omega^{x}_{ij}\right) \widetilde{\psi}^{\ell}_{jh}\right)$. In vector form are represented by

$$\ddot{\Lambda}^{G} = \left(\widetilde{\Omega}'_{\ell} + \widetilde{\Psi}'_{\ell} \left(\widetilde{\Omega}'_{x} - \Omega'_{x}\right)\right) \lambda = \widetilde{\Psi}'_{\ell} \left(I_N - \Omega'_{x}\right) \lambda.$$

From equation (107) we know that

$$\ddot{\Lambda}^{G} = \widetilde{\Lambda}.$$

- $\beta_{h|r} = \sum_{i\in\mathcal{N}_r} \beta_{hi}$ is household h's direct revenue intensity on goods produced by country

r.

- $\Omega^x_{j|r} = \sum_{i\in\mathscr{N}_r} \Omega^x_{ji}$ is firm j's direct revenue intensity on goods produced by country r.

Therefore

$$\begin{aligned}
\Phi_r\,\widehat{Y}_r = & \sum_{h\in\mathscr{H}} \chi_h \sum_{i\in\mathscr{N}_r} \beta_{hi}\,\widehat{\beta}_{hi} + \sum_{h\in\mathscr{H}} \beta_{h|r}\left(\Lambda_h\,\widehat{J}_h + \mathbb{T}_h\,\widehat{T}_h\right) \\
& + \sum_{h\in\mathscr{H}} \beta_{h|r} \sum_{i\in\mathscr{N}} \kappa_{ih}\,\lambda_i\left((1-\mu_i)\left(\widehat{\kappa}_{ih} + \widehat{S}_i\right) - \mu_i\,\widehat{\mu}_i\right) - \Phi_r \sum_{h\in\mathscr{H}} \ddot{\Lambda}^r_h\,\widehat{J}_h \\
& + \sum_{i\in\mathscr{N}} \Omega^x_{i|r}\,\lambda_i\left(\widehat{\omega}^x_i + \widehat{\mu}_i\right) - \sum_{q\in\mathscr{R}} \sum_{i\in\mathscr{N}_r} \Omega^x_{i|q}\,\lambda_i\left(\widehat{\omega}^x_i + \widehat{\mu}_i\right) \\
& + \sum_{i\notin\mathscr{N}_r} \lambda_i \sum_{j\in\mathscr{N}_r} \Omega^x_{ij}\,\widehat{\omega}_{ij} - \sum_{i\in\mathscr{N}_r} \lambda_i \sum_{j\notin\mathscr{N}_r} \Omega^x_{ij}\,\widehat{\omega}_{ij} + \Phi_r \sum_{j\in\mathscr{N}} \ddot{\lambda}^r_j\left(\widehat{\mathscr{A}}_j + \widehat{\mu}_j\right) \\
& + \sum_{i\in\mathscr{N}_r} \sum_{j\in\mathscr{N}} \Omega^x_{ji}\,\lambda_j\,\widehat{S}_j - \sum_{q\in\mathscr{R}} \sum_{i\in\mathscr{N}_r} \Omega^x_{i|q}\,\lambda_i\,\widehat{S}_i + \Phi_r \sum_{h\in\mathscr{H}} \ddot{\Lambda}^r_h\,\widehat{L}_h
\end{aligned} \tag{108}$$

As a consequence

$$Y_r = TFP_r\,F_r\left(\{L_h\}_{h\in\mathscr{H}}\right),$$

where $F_r\left(\{L_h\}_{h\in\mathscr{H}}\right)$ is a CRS function such that $\frac{d\log F_r\left(\{L_b\}_{b\in\mathscr{H}}\right)}{d\log L_h} = \ddot{\Lambda}^r_h$.

Add and subtract $\widehat{GDP}$ to express equation (108) in terms of sales and factor Domar weights

$$\begin{aligned}
\Phi_r\,\widehat{Y}_r = & \sum_{h\in\mathscr{H}} \chi_h \sum_{i\in\mathscr{N}_r} \beta_{hi}\,\widehat{\beta}_{hi} + \sum_{h\in\mathscr{H}} \beta_{h|r}\left(\Lambda_h\,\widehat{\Lambda}_h + \mathbb{T}_h\,\widehat{\mathbb{T}}_h\right) \\
& + \sum_{h\in\mathscr{H}} \beta_{h|r} \sum_{i\in\mathscr{N}} \kappa_{ih}\,\lambda_i\left((1-\mu_i)\left(\widehat{\kappa}_{ih} + \widehat{\lambda}_i\right) - \mu_i\,\widehat{\mu}_i\right) - \Phi_r \sum_{h\in\mathscr{H}} \ddot{\Lambda}^r_h\,\widehat{\Lambda}_h \\
& + \sum_{i\in\mathscr{N}} \Omega^x_{i|r}\,\lambda_i\left(\widehat{\omega}^x_i + \widehat{\mu}_i + \widehat{\lambda}_i\right) - \sum_{q\in\mathscr{R}} \sum_{i\in\mathscr{N}_r} \Omega^x_{i|q}\,\lambda_i\left(\widehat{\omega}^x_i + \widehat{\mu}_i + \widehat{\lambda}_i\right) \\
& + \sum_{i\notin\mathscr{N}_r} \lambda_i \sum_{j\in\mathscr{N}_r} \Omega^x_{ij}\,\widehat{\omega}_{ij} - \sum_{i\in\mathscr{N}_r} \lambda_i \sum_{j\notin\mathscr{N}_r} \Omega^x_{ij}\,\widehat{\omega}_{ij} + \Phi_r \sum_{j\in\mathscr{N}} \ddot{\lambda}^r_j\left(\widehat{\mathscr{A}}_j + \widehat{\mu}_j\right) + \Phi_r \sum_{h\in\mathscr{H}} \ddot{\Lambda}^r_h\,\widehat{L}_h \\
& + \left(\sum_{h\in\mathscr{H}} \beta_{h|r}\left(\Lambda_h + \sum_{i\in\mathscr{N}_q} \kappa_{ih}\,(1-\mu_i)\,\lambda_i + \mathbb{T}_h\right) + \sum_{i\in\mathscr{N}} \Omega^x_{j|r}\,\lambda_j - \sum_{q\in\mathscr{R}} \sum_{i\in\mathscr{N}_r} \Omega^x_{i|q}\,\lambda_i - 1\right)\widehat{GDP}
\end{aligned}$$

From equations (57), (75) and (100)

$$\begin{aligned}\Phi_r\,\widehat{Y}_r &= \sum_{h\in\mathscr{H}}\chi_h\sum_{i\in\mathscr{N}_r}\beta_{hi}\,\widehat{\beta}_{hi}+\sum_{h\in\mathscr{H}}\left(\beta_{h|r}\,\chi_h\,\widehat{\chi}_h-\Phi_r\,\ddot{\Lambda}_h^r\,\widehat{\Lambda}_h\right)\\ &+\sum_{i\in\mathscr{N}}\Omega_{i|r}^x\,\lambda_i\left(\widehat{\omega}_i^x+\widehat{\mu}_i+\widehat{\lambda}_i\right)-\sum_{q\in\mathscr{R}}\sum_{i\in\mathscr{N}_r}\Omega_{i|q}^x\,\lambda_i\left(\widehat{\omega}_i^x+\widehat{\mu}_i+\widehat{\lambda}_i\right)\\ &+\sum_{i\notin\mathscr{N}_r}\lambda_i\sum_{j\in\mathscr{N}_r}\Omega_{ij}^x\,\widehat{\omega}_{ij}-\sum_{i\in\mathscr{N}_r}\lambda_i\sum_{j\notin\mathscr{N}_r}\Omega_{ij}^x\,\widehat{\omega}_{ij}+\Phi_r\sum_{j\in\mathscr{N}}\ddot{\lambda}_j^r\left(\widehat{\mathscr{A}}_j+\widehat{\mu}_j\right)+\Phi_r\sum_{h\in\mathscr{H}}\ddot{\Lambda}_h^r\,\widehat{L}_h\\ &+\underbrace{\left(\sum_{h\in\mathscr{H}}\beta_{h|r}\,\chi_h+\sum_{i\in\mathscr{N}}\Omega_{j|r}^x\,\lambda_j-\sum_{q\in\mathscr{R}}\sum_{i\in\mathscr{N}_r}\Omega_{i|q}^x\,\lambda_i-1\right)}_{=0}\widehat{GDP}.\end{aligned}$$

The revenue distribution for country r is captured by the $H+N+dim\left(\mathscr{N}_r\right)$ vector

$$\mathscr{R}_r'=\Phi_r^{-1}\left[\left\{\beta_{h|r}\,\chi_h\right\}_{h\in\mathscr{H}}\quad\left\{\Omega_{i|r}^x\,\lambda_i\right\}_{i\in\mathscr{N}}\quad-\left\{\lambda_i\textstyle\sum_{q\in\mathscr{R}}\Omega_{i|q}^x\right\}_{i\in\mathscr{N}_r}\right].$$

Notice that as the elements of this vector add up to one, its first order approximation is given by

$$\begin{aligned}\Phi_r\,\widehat{\Phi}_r &= \sum_{h\in\mathscr{H}}\beta_{h|r}\,\chi_h\,\widehat{\chi}_h+\sum_{h\in\mathscr{H}}\chi_h\sum_{i\in\mathscr{N}_r}\beta_{hi}\,\widehat{\beta}_{hi}\\ &+\sum_{i\in\mathscr{N}}\Omega_{i|r}^x\,\lambda_i\left(\widehat{\omega}_i^x+\widehat{\mu}_i+\widehat{\lambda}_i\right)-\sum_{q\in\mathscr{R}}\sum_{i\in\mathscr{N}_r}\Omega_{i|q}^x\,\lambda_i\left(\widehat{\omega}_i^x+\widehat{\mu}_i+\widehat{\lambda}_i\right)\\ &+\sum_{i\notin\mathscr{N}_r}\lambda_i\sum_{j\in\mathscr{N}_r}\Omega_{ij}^x\,\widehat{\omega}_{ij}-\sum_{i\in\mathscr{N}_r}\lambda_i\sum_{j\notin\mathscr{N}_r}\Omega_{ij}^x\,\widehat{\omega}_{ij}.\end{aligned}$$

This implies that

$$\widehat{Y}_r=\sum_{j\in\mathscr{N}}\ddot{\lambda}_j^r\left(\widehat{\mathscr{A}}_j+\widehat{\mu}_j\right)+\widehat{\Phi}_r-\sum_{h\in\mathscr{H}}\ddot{\Lambda}_h^r\,\widehat{\Lambda}_h+\sum_{h\in\mathscr{H}}\ddot{\Lambda}_h^r\,\widehat{L}_h. \tag{109}$$

Now, using equations (70) and (93), and the definiitions $\delta_h^r=\frac{\ddot{\Lambda}_h^r}{\Lambda_h}$, $M_h^r=\sum_{b\in\mathscr{H}}\mathscr{C}_{hb}\,\delta_b^r$, and

$F_i^r = \sum_{h \in \mathscr{H}} \psi_{ih}^{\ell}\, \delta_h^r$

$$\begin{aligned}
\sum_{h \in \mathscr{H}} \ddot{\Lambda}_h^r\, \widehat{\Lambda}_h = \sum_{h \in \mathscr{H}} \delta_h^r\, d\,\Lambda_h& \\
&= \sum_{h \in \mathscr{H}} M_h^r\, d\,\chi_h + \sum_{h \in \mathscr{H}} \chi_h \sum_{b \in \mathscr{H}} \delta_b^r\, d\,\mathscr{C}_{hb} \\
&= \sum_{h \in \mathscr{H}} M_h^r\, d\,\chi_h + \sum_{i \in \mathscr{N}} \lambda_i\, F_i^r\, d\,log\,\mu_i + \sum_{i \in \mathscr{N}} \mu_i\, \lambda_i \sum_{h \in \mathscr{H}} \delta_h^r\, d\,\widetilde{\Omega}_{ih}^{\ell} \\
&+ \sum_{h \in \mathscr{H}} \chi_h \sum_{i \in \mathscr{N}} F_i^r\, d\,\beta_{hi} + \sum_{i \in \mathscr{N}} \mu_i\, \lambda_i \sum_{j \in \mathscr{N}} F_j^r\, d\,\widetilde{\Omega}_{ij}^{x}.
\end{aligned}$$

Then

$$\begin{aligned}
\widehat{TFP}_r &= \sum_{i \in \mathscr{N}} \ddot{\lambda}_i^r \left(\widehat{\mathscr{A}}_i + \widehat{\mu}_i\right) + \widehat{\Phi}_r - \sum_{h \in \mathscr{H}} M_h^r\, d\,\chi_h - \sum_{h \in \mathscr{H}} \chi_h \sum_{b \in \mathscr{H}} \delta_b^r\, d\,\mathscr{C}_{hb} \\
&= \sum_{i \in \mathscr{N}} \ddot{\lambda}_i^r \left(\widehat{\mathscr{A}}_i + \widehat{\mu}_i\right) + \widehat{\Phi}_r - \sum_{h \in \mathscr{H}} M_h^r\, d\,\chi_h - \sum_{i \in \mathscr{N}} \lambda_i\, F_i^r\, d\,log\,\mu_i - \sum_{i \in \mathscr{N}} \mu_i\, \lambda_i \sum_{h \in \mathscr{H}} \delta_h^r\, d\,\widetilde{\Omega}_{ih}^{\ell} \\
&- \sum_{h \in \mathscr{H}} \chi_h \sum_{i \in \mathscr{N}} F_i^r\, d\,\beta_{hi} - \sum_{i \in \mathscr{N}} \mu_i\, \lambda_i \sum_{j \in \mathscr{N}} F_j^r\, d\,\widetilde{\Omega}_{ij}^{x}.
\end{aligned} \tag{110}$$

2.6.6 Proof for Corollary 1 and Corollary 2

In the absence of distortions $\mu_i = 1\ \forall i \in \mathscr{N}$

$$\ddot{\lambda}_i^r = \mathbb{1}\left\{i \in \mathscr{N}_r\right\} \frac{\lambda_i}{\Phi_r}, \qquad \ddot{\Lambda}_h^r = \sum_{i \in \mathscr{N}_r} \widetilde{\Omega}_{ih}^{\ell} \frac{\lambda_i}{\Phi_r}, \qquad \Omega_{ij}^x = \widetilde{\Omega}_{ij}^x, \qquad \Omega_{ih}^{\ell} = \widetilde{\Omega}_{ih}^{\ell}.$$

Under these definitions

$$\begin{aligned}
\Phi_r\, \widehat{\Phi}_r = &\sum_{i \in \mathscr{N}_r} \sum_{h \in \mathscr{H}} \beta_{hi}\, \chi_h\, \widehat{\chi}_h + \sum_{h \in \mathscr{H}} \chi_h \sum_{i \in \mathscr{N}_r} \beta_{hi}\, \widehat{\beta}_{hi} \\
&+ \sum_{i \in \mathscr{N}_r} \sum_{j \in \mathscr{N}} \widetilde{\Omega}_{ji}^{x}\, \lambda_j \left(\widehat{\omega}_j^x + \widehat{\mu}_j + \widehat{\lambda}_j\right) - \sum_{i \in \mathscr{N}_r} \sum_{j \in \mathscr{N}} \widetilde{\Omega}_{ij}^{x}\, \lambda_i \left(\widehat{\omega}_i^x + \widehat{\mu}_i + \widehat{\lambda}_i\right) \\
&+ \sum_{i \notin \mathscr{N}_r} \lambda_i \sum_{j \in \mathscr{N}_r} \widetilde{\Omega}_{ij}^{x}\, \widehat{\omega}_{ij} - \sum_{i \in \mathscr{N}_r} \lambda_i \sum_{j \notin \mathscr{N}_r} \widetilde{\Omega}_{ij}^{x}\, \widehat{\omega}_{ij}.
\end{aligned}$$

and equation (109) is given by

$$\Phi_r\, \widehat{TFP}_r = \sum_{i \in \mathscr{N}_r} \lambda_i \left(\widehat{\mathscr{A}}_i + \widehat{\mu}_i\right) + \Phi_r\, \widehat{\Phi}_r - \sum_{i \in \mathscr{N}_r} \lambda_i \sum_{h \in \mathscr{H}} \widetilde{\Omega}_{ih}^{\ell} \widehat{\Lambda}_h.$$

Without distortions $GDP_r = GNI_r \equiv \sum_{h\in\mathscr{H}_r} E_h$. From equation (103) this implies that

$$\Phi_r\,\widehat{\Phi}_r = \Phi_r^{GNI}\,\widehat{\Phi}_r^{GNI} = \sum_{h\in\mathscr{H}_r}\left(\Lambda_h\,\widehat{\Lambda}_h - \sum_{i\in\mathscr{N}}\kappa_{ih}\,\lambda_i\,\widehat{\mu}_i\right).$$

Thus

$$\Phi_r\,\widehat{TFP}_r = \sum_{i\in\mathscr{N}_r}\lambda_i\,\widehat{\mathscr{A}}_i + \sum_{i\in\mathscr{N}_r}\lambda_i\left(1-\sum_{h\in\mathscr{H}_r}\kappa_{ih}\right)\widehat{\mu}_i - \sum_{h\in\mathscr{H}_r}\sum_{i\notin\mathscr{N}_r}\kappa_{ih}\,\lambda_i\,\widehat{\mu}_i$$
$$+\sum_{h\in\mathscr{H}_r}\left(\Lambda_h - \sum_{i\in\mathscr{N}_r}\widetilde{\Omega}^{\ell}_{ih}\,\lambda_i\right)\widehat{\Lambda}_h - \sum_{i\in\mathscr{N}_r}\lambda_i\sum_{h\notin\mathscr{H}_r}\widetilde{\Omega}^{\ell}_{ih}\widehat{\Lambda}_h.$$

Assuming country-specific labor markets and total equity home bias, we know that

$$\sum_{i\in\mathscr{N}_r}\widetilde{\Omega}^{\ell}_{ih}\lambda_i = \begin{cases}\Lambda_h & \text{if} \quad h\in\mathscr{H}_r \\ 0 & \text{otherwise}\end{cases}, \qquad \sum_{h\in\mathscr{H}_r}\kappa_{ih} = \begin{cases}1 & \text{if} \quad i\in\mathscr{N}_r \\ 0 & \text{otherwise}\end{cases}.$$

Hence

$$\Phi_r\,\widehat{TFP}_r = \sum_{i\in\mathscr{N}_r}\lambda_i\,\widehat{\mathscr{A}}_i.$$

2.6.7 Proof for Theorem 3

Global GDP is given by

$$GDP = \sum_{i\in\mathscr{N}}\left(1-\mu_i\sum_{j\in\mathscr{N}}\widetilde{\Omega}^{x}_{ij}\right)S_i.$$

Hence, following the same steps as in Section 2.6.5

$$GDP\,\widehat{GDP} = \sum_{r\in\mathscr{R}}\sum_{i\in\mathscr{N}_r}S_i\left(\widehat{S}_i - \sum_{j\in\mathscr{N}}\mu_i\,\omega^{x}_i\,\omega_{ij}\left(\widehat{\mu}_i + \widehat{\omega}^{x}_i + \widehat{\omega}_{ij} + \widehat{S}_i\right)\right).$$

From equations (30) and (87)

$$\widehat{GDP} = \sum_{i\in\mathscr{N}}\lambda_i\left(\widehat{S}_i - \sum_{j\in\mathscr{N}}\Omega^{x}_{ij}\left(\widehat{p}_j + \widehat{x}_{ij}\right)\right).$$

From here, I define the first-order approximation for the global GDP deflator as

$$\widehat{p}_Y = \sum_{i\in\mathscr{N}} \lambda_i \left(\widehat{p}_i - \sum_{j\in\mathscr{N}} \Omega^x_{ij}\, \widehat{p}_j\right).$$

Hence, the first-order approximation for global real GDP is given by

$$\widehat{Y} = \sum_{r\in\mathscr{R}} \Phi_r\, \widehat{Y}_r = \sum_{i\in\mathscr{N}} \lambda_i \left(\widehat{y}_i - \sum_{j\in\mathscr{N}} \Omega^x_{ij}\, \widehat{x}_{ij}\right).$$

Using the first-order approximation for the goods market clearing condition

$$\widehat{Y} = \sum_{i\in\mathscr{N}} \left(\sum_{h\in\mathscr{H}} \beta_{hi}\, \chi_h\, \widehat{C}_{hi} + \sum_{j\in\mathscr{N}} \left(\Omega^x_{ji}\, \lambda_j\, \widehat{x}_{ji} - \Omega^x_{ij}\, \lambda_i\, \widehat{x}_{ij}\right)\right) = \sum_{h\in\mathscr{H}} \chi_h \sum_{i\in\mathscr{N}} \frac{\beta_{hi}}{\tau^c_{hi}} \widehat{C}_{hi}.$$

Introducing the first-order approximation of equations (46), (57), (90), and (96)

$$\widehat{Y} = \sum_{h\in\mathscr{H}} \chi_h \sum_{i\in\mathscr{N}} \beta_{hi}\widehat{\beta}_{hi} + \sum_{h\in\mathscr{H}} \left(\sum_{i\in\mathscr{N}} \beta_{hi}\right) \widehat{E}_h - \sum_{i\in\mathscr{N}} \underbrace{\left(\sum_{h\in\mathscr{H}} \beta_{hi}\, \chi_h\right)}_{=\lambda_i - \sum_{j\in\mathscr{N}} \Omega^x_{ji}\, \lambda_j} \widehat{p}_i.$$

$$\widehat{Y} = \sum_{h\in\mathscr{H}} \chi_h \sum_{i\in\mathscr{N}} \beta_{hi}\, \widehat{\beta}_{hi} + \sum_{h\in\mathscr{H}} \left(\sum_{i\in\mathscr{N}} \beta_{hi}\right) \left(\Lambda_h \left(\widehat{w}_h + \widehat{L}_h\right) + \mathbb{T}_h\, \widehat{T}_h\right) + \sum_{i\in\mathscr{N}} \sum_{j\in\mathscr{N}} \Omega^x_{ij}\, \lambda_i\, \widehat{p}_j - \sum_{i\in\mathscr{N}} \lambda_i\, \widehat{p}_i$$
$$+ \sum_{h\in\mathscr{H}} \left(\sum_{j\in\mathscr{N}} \beta_{hj}\right) \sum_{i\in\mathscr{N}_q} \kappa_{ih}\, \lambda_i \left((1-\mu_i)\left(\widehat{\kappa}_{ih} + \widehat{S}_i\right) - \mu_i\, \widehat{\mu}_i\right).$$

$$\begin{aligned}
\widehat{Y} = {} & \sum_{h\in\mathscr{H}} \chi_h \sum_{i\in\mathscr{N}} \beta_{hi}\, \widehat{\beta}_{hi} + \sum_{h\in\mathscr{H}} \left(\sum_{i\in\mathscr{N}} \beta_{hi}\right) \left(\Lambda_h\, \widehat{J}_h + \mathbb{T}_h\, \widehat{T}_h\right) \\
& + \sum_{h\in\mathscr{H}} \left(\sum_{j\in\mathscr{N}} \beta_{hj}\right) \sum_{i\in\mathscr{N}} \kappa_{ih}\, \lambda_i \left((1-\mu_i)\left(\widehat{\kappa}_{ih} + \widehat{S}_i\right) - \mu_i\, \widehat{\mu}_i\right) \\
& + \sum_{j\in\mathscr{N}} \sum_{i\in\mathscr{N}} \lambda_i \left(\widetilde{\psi}^x_{ij} - \sum_{m\in\mathscr{N}} \Omega^x_{im}\, \widetilde{\psi}^x_{mj}\right) \left(\widehat{\mathscr{A}}_j + \widehat{\mu}_j\right) \\
& - \sum_{h\in\mathscr{H}} \sum_{r\in\mathscr{R}} \left(\sum_{i\in\mathscr{N}_r} \lambda_i \left(\widetilde{\psi}^\ell_{ih} - \sum_{j\in\mathscr{N}} \Omega^x_{ij}\, \widetilde{\psi}^\ell_{jh}\right)\right) \widehat{J}_h \\
& + \sum_{h\in\mathscr{H}} \left(\sum_{i\in\mathscr{N}} \lambda_i \left(\widetilde{\psi}^\ell_{ih} - \sum_{j\in\mathscr{N}} \Omega^x_{ij}\, \widetilde{\psi}^\ell_{jh}\right)\right) \widehat{L}_h.
\end{aligned}$$

Then

$$
\begin{aligned}
\widehat{Y} = & \sum_{h\in\mathscr{H}} \chi_h \sum_{i\in\mathscr{N}} \beta_{hi}\, \widehat{\beta}_{hi} + \sum_{h\in\mathscr{H}} \left(\sum_{i\in\mathscr{N}} \beta_{hi}\right) \left(\Lambda_h\, \widehat{J}_h + \mathbb{T}_h\, \widehat{T}_h\right) \\
& + \sum_{h\in\mathscr{H}} \left(\sum_{j\in\mathscr{N}} \beta_{hj}\right) \sum_{i\in\mathscr{N}} \kappa_{ih}\, \lambda_i \left((1-\mu_i)\left(\widehat{\kappa}_{ih} + \widehat{S}_i\right) - \mu_i\, \widehat{\mu}_i\right) \\
& + \sum_{i\in\mathscr{N}} \ddot{\lambda}_i^G \left(\widehat{\mathscr{A}}_i + \widehat{\mu}_i\right) - \sum_{h\in\mathscr{H}} \sum_{r\in\mathscr{R}} \Phi_r\, \ddot{\Lambda}_h^r\, \widehat{J}_h + \sum_{h\in\mathscr{H}} \ddot{\Lambda}_h^G\, \widehat{L}_h.
\end{aligned}
\tag{111}
$$

As a consequence

$$
Y = TFP\, F\left(\{L_h\}_{h\in\mathscr{H}}\right),
$$

where $F\left(\{L_h\}_{h\in\mathscr{H}}\right)$ is a CRS function such that $\frac{d\log F\left(\{L_b\}_{b\in\mathscr{H}}\right)}{d\log L_h} = \ddot{\Lambda}_h^G$.

Add and subtract $\widehat{GDP}$ to express equation (111) in terms of sales and factor Domar weights

$$
\begin{aligned}
\widehat{Y} = & \sum_{h\in\mathscr{H}} \chi_h \sum_{i\in\mathscr{N}} \beta_{hi}\, \widehat{\beta}_{hi} + \sum_{h\in\mathscr{H}} \left(\sum_{i\in\mathscr{N}} \beta_{hi}\right) \underbrace{\left(\Lambda_h\, \widehat{\Lambda}_h + \mathbb{T}_h\, \widehat{\mathbb{T}}_h + \sum_{i\in\mathscr{N}} \kappa_{ih}\, \lambda_i \left((1-\mu_i)\left(\widehat{\kappa}_{ih} + \widehat{\lambda}_i\right) - \mu_i\, \widehat{\mu}_i\right)\right)}_{=\chi_h\, \widehat{\chi}_h} \\
& + \sum_{i\in\mathscr{N}} \ddot{\lambda}_i^G \left(\widehat{\mathscr{A}}_i + \widehat{\mu}_i\right) - \sum_{h\in\mathscr{H}} \ddot{\Lambda}_h^G\, \widehat{\Lambda}_h + \sum_{h\in\mathscr{H}} \ddot{\Lambda}_h^G\, \widehat{L}_h \\
& + \left(\overbrace{\sum_{i\in\mathscr{N}} \underbrace{\sum_{h\in\mathscr{H}} (\beta_{hi}) \underbrace{\left(\Lambda_h + \sum_{i\in\mathscr{N}_q} \kappa_{ih}\,(1-\mu_i)\,\lambda_i + \mathbb{T}_h\right)}_{=\chi_h}}_{=\lambda_i - \sum_{j\in\mathscr{N}} \Omega^x_{ji}\lambda_j}}^{=\sum_{i\in\mathscr{N}}(1-\sum_{j\in\mathscr{N}}\Omega_{ij})\lambda_i = 1} - \overbrace{\sum_{r\in\mathscr{R}} \Phi_r \underbrace{\sum_{h\in\mathscr{H}} \ddot{\Lambda}_h^r}_{=1}}^{=1} \right) \widehat{GDP}.
\end{aligned}
$$

From equation (100)

$$
\widehat{Y} = \sum_{i\in\mathscr{N}} \ddot{\lambda}_i^G \left(\widehat{\mathscr{A}}_i + \widehat{\mu}_i\right) + \sum_{i\in\mathscr{N}} \sum_{h\in\mathscr{H}} \beta_{hi}\, \chi_h \left(\widehat{\chi}_h + \widehat{\beta}_{hi}\right) - \sum_{h\in\mathscr{H}} \ddot{\Lambda}_h^G\, \widehat{\Lambda}_h + \sum_{h\in\mathscr{H}} \ddot{\Lambda}_h^G\, \widehat{L}_h.
$$

The global distribution of sectoral revenue from final sales is captured by the N vector

$$
\mathscr{R} = \left[\left\{\sum_{h\in\mathscr{H}} \beta_{hi}\, \chi_h\right\}_{i\in\mathscr{N}}\right].
$$

Notice that as $\sum_{i\in\mathscr{N}}\sum_{h\in\mathscr{H}}\beta_{hi}\,\chi_h = \sum_{i\in\mathscr{N}}\left(1-\sum_{j\in\mathscr{N}}\Omega_{ij}\right)\lambda_i = 1$, then

$$\sum_{i\in\mathscr{N}}\sum_{h\in\mathscr{H}}\beta_{hi}\,\chi_h\left(\widehat{\chi}_h+\widehat{\beta}_{hi}\right)=0.$$

This implies that

$$\widehat{Y}=\sum_{i\in\mathscr{N}}\widetilde{\lambda}_i\left(\widehat{\mathscr{A}}_i+\widehat{\mu}_i\right)-\sum_{h\in\mathscr{H}}\widetilde{\Lambda}_h\,\widehat{\Lambda}_h+\sum_{h\in\mathscr{H}}\widetilde{\Lambda}_h\,\widehat{L}_h.$$

Add and subtract $\widehat{\xi}_{1r}$ to guarantee that the previous equation is represented in terms of exchange rate between all countries and country 1

$$\widehat{Y}=\sum_{i\in\mathscr{N}}\widetilde{\lambda}_i\left(\widehat{\mathscr{A}}_i+\widehat{\mu}_i\right)-\sum_{h\in\mathscr{H}}\widetilde{\Lambda}_h\,\widehat{\Lambda}_h+\sum_{h\in\mathscr{H}}\widetilde{\Lambda}_h\,\widehat{L}_h. \tag{112}$$

Now, using equations (70) and (93), and the definiitions $\delta_h=\frac{\widetilde{\Lambda}_h}{\Lambda_h}$, $M_h=\sum_{b\in\mathscr{H}}\mathscr{C}_{hb}\,\delta_b$, and $F_i=\sum_{h\in\mathscr{H}}\psi^{\ell}_{ih}\,\delta_h$

$$\begin{aligned}\sum_{h\in\mathscr{H}}\widetilde{\Lambda}_h\,\widehat{\Lambda}_h&=\sum_{h\in\mathscr{H}}\delta_h\,d\,\Lambda_h\\&=\sum_{h\in\mathscr{H}}M_h\,d\,\chi_h+\sum_{h\in\mathscr{H}}\chi_h\sum_{b\in\mathscr{H}}\delta_b\,d\,\mathscr{C}_{hb}\\&=\sum_{h\in\mathscr{H}}M_h\,d\,\chi_h+\sum_{i\in\mathscr{N}}\lambda_i\,F_i\,d\,log\,\mu_i+\sum_{i\in\mathscr{N}}\mu_i\,\lambda_i\sum_{h\in\mathscr{H}}\delta_h\,d\,\widetilde{\Omega}^{\ell}_{ih}\\&+\sum_{h\in\mathscr{H}}\chi_h\sum_{i\in\mathscr{N}}F_i\,d\,\beta_{hi}+\sum_{i\in\mathscr{N}}\mu_i\,\lambda_i\sum_{j\in\mathscr{N}}F_j\,d\,\widetilde{\Omega}^{x}_{ij}.\end{aligned} \tag{113}$$

2.6.8 Proof for Hulten (1978)

Without distortions equation (112) is given by

$$\widehat{Y}=\sum_{i\in\mathscr{N}}\lambda_i\left(\widehat{\mathscr{A}}_i+\widehat{\mu}_i\right)-\sum_{h\in\mathscr{H}}\Lambda_h\,\widehat{\Lambda}_h+\sum_{h\in\mathscr{H}}\Lambda_h\,\widehat{L}_h.$$

Additionally, $\delta_h=M_h=F_i=1\ \forall h\in\mathscr{H}$ and $\forall i\in\mathscr{N}$. Equation (113) is given by

$$\begin{aligned}\sum_{h\in\mathscr{H}}\Lambda_h\,\widehat{\Lambda}_h&=\sum_{h\in\mathscr{H}}d\,\chi_h+\sum_{i\in\mathscr{N}}\lambda_i\,d\,log\,\mu_i+\sum_{i\in\mathscr{N}}\lambda_i\left(d\,\omega^{\ell}_i+d\,\omega^{x}_i\right)+\sum_{h\in\mathscr{H}}\chi_h\sum_{i\in\mathscr{N}}d\,\beta_{hi}\\&=\sum_{i\in\mathscr{N}}\lambda_i\,d\,log\,\mu_i.\end{aligned}$$

This implies that

$$\widehat{TFP} = \sum_{i \in \mathscr{N}} \lambda_i \, \widehat{\mathscr{A}}_i.$$

3 Relationship with Baqaee & Farhi (2020) and Baqaee & Farhi (2023)

3.1 The Environment

Baqaee & Farhi (2023) introduces an open-economy model with production networks, heterogeneous households, and firm level distortions captured by wedges between marginal productivities and prices. Baqaee & Farhi (2020) is a particular case of Baqaee & Farhi (2023) with a closed-economy and a representative household. Their model is characterized by:

- A set of countries $\mathscr{R}$, a set of producers of different goods $\mathscr{N}$, and a set of factors $\mathscr{L}$;
- The set of producers that operate within the borders of country $r \in \mathscr{R}$ are $\mathscr{N}_r \subseteq \mathscr{N}$;
- The set of factors that are exclusively used by firms in country $r \in \mathscr{R}$ are $\mathscr{L}_r \subseteq \mathscr{L}$;
- Each country has a representative household, for which the ownership of primary factors $\mathscr{L}$ and fictitious factors $\mathscr{L}^*$ (fictitious factors represent the collection of markup/wedge revenue rebated back to households) is characterized by the $R \times (L + L^*)$ ownership matrix Φ, where Φ_{ri} captures the share of factor i's added value that is a source of income for household r;
- Primary factor supply is exogenous;

Producer $i \in \mathscr{N}_r$ uses a CRS production function

$$y_i = A_i Q_i \left(\{\ell_{il}\}_{l \in \mathscr{L}_c}, \{x_{ij}\}_{j \in \mathscr{N}} \right),$$

with $mc_i = \mu_i \times p_i$, i.e. marginal cost equals price times a markdown.[4]

The representative household for country $r \in \mathscr{R}$ has homothetic preferences given by the consumption aggregator[5]

$$C_r = Q_r^c \left(\{C_{ri}\}_{i \in \mathscr{N}} \right),$$

[4]In their model wedges are represented by a markup, but here I use the inverse of the markdown both for algebraic simplicity and to maintain notation equivalence with my model.

[5]In their model this real consumption aggregator is used as a measure of welfare at the country level.

and faces a budget constraint given by

$$\sum_{i\in\mathscr{N}} p_i C_{ri} = \sum_{l\in\mathscr{L}} \Phi_{rl} w_l L_l + \sum_{i\in\mathscr{N}} \Phi_{ri}\left(1-\mu_i\right) p_i y_i + T_r.$$

Equilibrium market clearing conditions are given by $y_i = \sum_{r\in\mathscr{R}} C_{ri} + \sum_{j\in\mathscr{N}} x_{ji}\ \forall i \in \mathscr{N}$, $L_l = \sum_{i\in\mathscr{N}_r} \ell_{il}\ \forall l \in \mathscr{L}_r$, and $\sum_{r\in\mathscr{R}} T_r = 0$.

Their main contribution is to show how the first-order approximation to firm level productivity and distortions shocks for real GDP and welfare (understand real consumption) at the country and global level can be decomposed into a direct technology effect, and a pure reallocation component.

3.2 Main structural differences

The main structural differences between Baqaee & Farhi's 2023 and my environment are:

1. In their model, factor markets are segmented by country, which as it will be shown above, allows them to represent country level GDP in term of factoral income. In my model, I am agnostic about the geographical segmentation of factor markets, and for this reason, country specific factor markets are a particular case in which $\alpha_{ih} > 0$ is allowed to occur only for $i \in \mathscr{N}_r$ and $h \in \mathscr{H}_r$.

2. In their model, each country representative household owns a portfolio of primary and fictitious factors, and primary factor supply is assumed as exogenous. In my model, the set of households that reside within the border of the country $r \in \mathscr{R}$ are $\mathscr{H}_r \subseteq \mathscr{H}$, and each type of household $h \in \mathscr{H}_r$ supplies only one type of primary factor that can be used by any firm across the globe, while the matrix κ describes the distribution across households of profits generated by markdown revenue that they denominate fictitious factor income.

   The additional gain in my model from restricting the supply of each household to only one type of primary factor is that this primary factor supply is endogenous. This allows me to decompose the distributional sources of variation using the first-order approximation for the labor wedges.

   To be fair, the model in their appendix allows for heterogeneous households within countries, and an endogenous labor-leisure tradeoff. But their endogeneity in primary factor supply is not microfounded as in my model, but rather assumed, and restricted to an elastic positive response to real wages and a negative response to real income.

3. Finally, in their model normalization is done using global nominal GDP as the numeraire,

and this normalization is required both for the ex-post and ex-ante first-order approximations. In my model, normalization is only required for the ex-ante first-order approximations and in these cases, the more standard real GDP as the numeraire is used.

In principle their normalization assumption appears as inconsequential, but as soon as one starts to wonder about the implications of normalizing with $GDP = 1$, the natural question that follows is about the real unit of account that acts as numeraire. Assume two scenarios for global real GDP, in the first one $Y = 1$, and in the second one $Y = 2$. Normalizing with $GDP = 1$ implies for the GDP deflator in the first case that $P_Y = 1$, and in the second one $P_Y = 1/2$. Additionally, for an economy with no intermediate input consumption, assume that a commodity M with an exogenous supply and a price V is used as a medium of exchange, and this commodity also acts as the real unit of account, therefore $VM = GDP$ in equilibrium and $V = 1$. Assume that the supply for M is increased, the assumption that $GDP = 1$ implies an excess supply for the medium of exchange M and as consequence V has to fall. Therefore when normalization is done with nominal GDP there is no real unit of account that acts as numeraire.

3.3 Notational equivalences

Let me start by defining the net quantity of good $i \in \mathscr{N}$ produced by country $r \in \mathscr{R}$

$$q_{ri} = y_i \mathbb{1}\left\{i \in \mathscr{N}_r\right\} - \sum_{j \in \mathscr{N}_r} x_{ji} = \begin{cases} y_i - \sum_{j \in \mathscr{N}_r} x_{ji} & \text{if} \quad i \in \mathscr{N}_r \\ -\sum_{j \in \mathscr{N}_r} x_{ji} & \text{if} \quad i \notin \mathscr{N}_r \end{cases} .$$

From here, the share of q_{ri} in the final output of country r is given by

$$\Omega_{Y_r i} = \frac{p_i q_{ri}}{GDP_r}.$$

Now, from equation (105)

$$
\begin{aligned}
\widehat{p}_{Y_r} &= \sum_{i\in\mathcal{N}_r} \frac{\lambda_i}{\Phi_r}\left(\widehat{p}_i - \sum_{j\in\mathcal{N}} \Omega^x_{ij}\widehat{p}_j\right) \\
&= \sum_{i\in\mathcal{N}_r} \frac{\lambda_i}{\Phi_r}\widehat{p}_i - \sum_{i\in\mathcal{N}_r} \frac{\lambda_i}{\Phi_r} \sum_{j\in\mathcal{N}_r} \Omega^x_{ij}\,\widehat{p}_j - \sum_{i\in\mathcal{N}_r} \frac{\lambda_i}{\Phi_r} \sum_{j\notin\mathcal{N}_r} \Omega^x_{ij}\,\widehat{p}_j \\
&= \sum_{i\in\mathcal{N}_r} \left(\frac{\lambda_i}{\Phi_r} - \sum_{j\in\mathcal{N}_r} \Omega^x_{ji}\frac{\lambda_j}{\Phi_r}\right)\widehat{p}_i - \sum_{i\notin\mathcal{N}_r}\left(\sum_{j\in\mathcal{N}_r} \Omega^x_{ji}\frac{\lambda_j}{\Phi_r}\right)\widehat{p}_i \\
&= \sum_{i\in\mathcal{N}_r} \left(\frac{S_i}{GDP_r} - \sum_{j\in\mathcal{N}_r} \frac{p_i x_{ji}}{S_j}\frac{S_j}{GDP_r}\right)\widehat{p}_i - \sum_{i\notin\mathcal{N}_r}\left(\sum_{j\in\mathcal{N}_r} \frac{p_i x_{ji}}{S_j}\frac{S_j}{GDP_r}\right)\widehat{p}_i \\
&= \sum_{i\in\mathcal{N}_r} \frac{p_i}{GDP_r}\left(y_i - \sum_{j\in\mathcal{N}_r} x_{ji}\right)\widehat{p}_i - \sum_{i\notin\mathcal{N}_r} \frac{p_i}{GDP_r}\left(\sum_{j\in\mathcal{N}_r} x_{ji}\right)\widehat{p}_i = \sum_{i\in\mathcal{N}} \Omega_{Y_r i}\,\widehat{p}_i.
\end{aligned}
$$

This matches the country-level Divisia index GDP deflator used by Baqaee & Farhi (2023).

Now, from equation (106)

$$
\begin{aligned}
\widehat{Y}_r &= \sum_{i\in\mathcal{N}_r} \frac{\lambda_i}{\Phi_r}\left(\widehat{y}_i - \sum_{j\in\mathcal{N}} \Omega^x_{ij}\widehat{x}_{ij}\right) \\
&= \sum_{i\in\mathcal{N}_r} \left(\frac{\lambda_i}{\Phi_r}\widehat{y}_i - \sum_{j\in\mathcal{N}_r} \Omega^x_{ji}\frac{\lambda_j}{\Phi_r}\widehat{x}_{ji}\right) - \sum_{i\notin\mathcal{N}_r}\sum_{j\in\mathcal{N}_r} \Omega^x_{ji}\frac{\lambda_j}{\Phi_r}\widehat{x}_{ji} \\
&= \sum_{i\in\mathcal{N}_r} \Omega_{Y_r i}\left(\frac{S_i}{p_i q_{ri}}\widehat{y}_i - \sum_{j\in\mathcal{N}_r} \frac{p_i x_{ji}}{p_i q_{ri}}\widehat{x}_{ji}\right) - \sum_{i\notin\mathcal{N}_r} \Omega_{Y_r i}\sum_{j\in\mathcal{N}_r} \frac{p_i x_{ji}}{p_i q_{ri}}\widehat{x}_{ji} \\
&= \sum_{i\in\mathcal{N}_r} \Omega_{Y_r i}\widehat{q}_{ri} - \sum_{i\notin\mathcal{N}_r} \Omega_{Y_r i}\widehat{q}_{ri} = \sum_{i\in\mathcal{N}} \Omega_{Y_r i}\widehat{q}_{ri}.
\end{aligned}
$$

This matches the country-level Divisia index real GDP variation used by Baqaee & Farhi (2023).

3.4 First order approximation for country-level GDP

Notice that one essential difference between my approach towards the first order approximation for country-level GDP that leads to equation (109) is that my proof starts from the expenditure definition of GDP in equation (75), i.e. the sum of the final uses of goods and services in purchaser prices, excluding intermediate input consumption, for firms that operate in the country. On the other hand, Baqaee & Farhi's (2023) proof starts from the income definition of GDP in equation (76), i.e. the sum of factor compensation and corporate profits distributed by the firms that produce in the country.

From equation (76), the first order approximation for the income definition of nominal GDP is given by

$$\widehat{GDP}_r = \widehat{p}_{Y_r} + \widehat{Y}_r = \sum_{i \in \mathscr{N}_r} \frac{\lambda_i}{\Phi_r} \sum_{h \in \mathscr{H}} \Omega^{\ell}_{ih} \left(\widehat{w}_h + \widehat{\ell}_{ih} \right) + \sum_{i \in \mathscr{N}_r} \frac{\lambda_i}{\Phi_r} \left((1-\mu_i) \widehat{S}_i - \mu_i \widehat{\mu}_i \right).$$

Now, starting from equation (88)

$$\widehat{p}_{i \in \mathscr{N}_r} = \left(I_{N_r} - \widetilde{\Omega}^{D_r}_x \right)^{-1} \left(\widetilde{\Omega}^r_{\ell} \widehat{w} + \widetilde{\Omega}^{M_r}_x \widehat{p}_{i \notin \mathscr{N}_r} - \widehat{A} - \widehat{\mu} \right),$$

where $\widetilde{\Omega}^{D_r}_x$ is the $N_r \times N_r$ domestic cost-based input-output matrix, $\widetilde{\Omega}^{M_r}_x$ is the $N_r \times (N - N_r)$ imported cost-based input-output matrix, $\widetilde{\Omega}^r_{\ell}$ is the $N_r \times H$ domestic cost-based factor matrix, $\widehat{p}_{i \in \mathscr{N}_r}$ is a vector of dimension N_r that captures the variation for domestic prices, and $\widehat{p}_{i \notin \mathscr{N}_r}$ is a vector of dimension $N - N_r$ that captures the variation for foreign prices. Notice that $\widetilde{\Omega}^{D_r}_x$ and $\widetilde{\Omega}^{M_r}_x$ are coming from a reorganization of the rows in $\widetilde{\Omega}_x$ that characterize the intermediate input demand for firms that operate in country r, and $\widetilde{\Omega}^r_{\ell}$ is composed of the rows in $\widetilde{\Omega}_{\ell}$ that characterize the primary factor demand for firms that operate in country r. Let me introduce the following definitions that come from Baqaee & Farhi (2023)

- $\widetilde{\psi}^{x_r}_{ij}$ represents the ij element of matrix $\left(I_{N_r} - \widetilde{\Omega}^{D_r}_x \right)^{-1}$;
- $\widetilde{\lambda}_{Y_r j} = \sum_{i \in \mathscr{N}_r} \Omega_{Y_r i} \widetilde{\psi}^{x_r}_{ij}$;
- $\widetilde{\Lambda}_{Y_r h} = \sum_{i \in \mathscr{N}_r} \Omega_{Y_r i} \sum_{j \in \mathscr{N}_r} \widetilde{\psi}^{x_r}_{ij} \widetilde{\Omega}^{\ell}_{jh}$ for $h \in \mathscr{H}$;
- $\widetilde{\Lambda}_{Y_r i} = \sum_{m \in \mathscr{N}_r} \Omega_{Y_r m} \sum_{j \in \mathscr{N}_r} \widetilde{\psi}^{x_r}_{mj} \widetilde{\Omega}^{x}_{ji}$ for $i \notin \mathscr{N}_r$;
- $J_{ih} = w_h \ell_{ih}$;
- $\Lambda_{Y_r h} = \frac{\sum_{i \in \mathscr{N}_r} J_{ih}}{GDP_r}$ for $h \in \mathscr{H}$;
- $\lambda_{Y_r i} = \frac{p_i y_i}{GDP_r}$ for $i \in \mathscr{N}_r$.

Now, introducing equation (90) in equation (105)

$$\begin{aligned}
\widehat{p}_{Y_r} &= \sum_{i \in \mathscr{N}} \Omega_{Y_r i} \widehat{p}_i = \sum_{i \in \mathscr{N}_r} \Omega_{Y_r i} \widehat{p}_i + \sum_{i \notin \mathscr{N}_r} \Omega_{Y_r i} \widehat{p}_i \\
&= \sum_{i \in \mathscr{N}_r} \Omega_{Y_r i} \left(\sum_{j \in \mathscr{N}_r} \widetilde{\psi}^{x_r}_{ij} \sum_{h \in \mathscr{H}} \widetilde{\Omega}^{\ell}_{jh} \widehat{w}_h + \sum_{j \in \mathscr{N}_r} \widetilde{\psi}^{x_r}_{ij} \sum_{i \notin \mathscr{N}_r} \widetilde{\Omega}^{x}_{ji} \widehat{p}_i - \sum_{j \in \mathscr{N}_r} \widetilde{\psi}^{x}_{ij} \left(\widehat{\mathscr{A}}_j + \widehat{\mu}_j \right) \right) + \sum_{i \notin \mathscr{N}_r} \Omega_{Y_r i} \widehat{p}_i \\
&= \sum_{h \in \mathscr{H}} \widetilde{\Lambda}_{Y_r h} \widehat{w}_h - \sum_{j \in \mathscr{N}_r} \widetilde{\lambda}_{Y_r j} \left(\widehat{\mathscr{A}}_j + \widehat{\mu}_j \right) + \sum_{i \notin \mathscr{N}_r} \widetilde{\Lambda}_{Y_r i} \widehat{p}_i + \sum_{i \notin \mathscr{N}_r} \Omega_{Y_r i} \widehat{p}_i.
\end{aligned}$$

Then

$$
\begin{aligned}
\widehat{Y}_r &= \sum_{i\in\mathscr{N}_r}\frac{\lambda_i}{\Phi_r}\sum_{h\in\mathscr{H}}\Omega^{\ell}_{ih}\left(\widehat{w}_h+\widehat{\ell}_{ih}\right)+\sum_{i\in\mathscr{N}_r}\frac{\lambda_i}{\Phi_r}\left(\left(1-\mu_i\right)\widehat{S}_i-\mu_i\widehat{\mu}_i\right)-\widehat{p}_{Y_r}\\
&=\sum_{i\in\mathscr{N}_r}\frac{\lambda_i}{\Phi_r}\sum_{h\in\mathscr{H}}\Omega^{\ell}_{ih}\left(\widehat{w}_h+\widehat{\ell}_{ih}\right)+\sum_{i\in\mathscr{N}_r}\frac{\lambda_i}{\Phi_r}\left(\left(1-\mu_i\right)\widehat{S}_i-\mu_i\widehat{\mu}_i\right)\\
&-\sum_{h\in\mathscr{H}}\widetilde{\Lambda}_{Y_rh}\widehat{w}_h+\sum_{j\in\mathscr{N}_r}\widetilde{\lambda}_{Y_rj}\left(\widehat{\mathscr{A}}_j+\widehat{\mu}_j\right)-\sum_{i\notin\mathscr{N}_r}\widetilde{\Lambda}_{Y_ri}\widehat{p}_i-\sum_{i\notin\mathscr{N}_r}\Omega_{Y_ri}\,\widehat{p}_i\\
&=\sum_{i\in\mathscr{N}_r}\frac{\lambda_i}{\Phi_r}\sum_{h\in\mathscr{H}}\Omega^{\ell}_{ih}\left(\widehat{w}_h+\widehat{\ell}_{fh}\right)+\sum_{i\in\mathscr{N}_r}\frac{\lambda_i}{\Phi_r}\left(\left(1-\mu_i\right)\widehat{S}_i-\mu_i\widehat{\mu}_i\right)\\
&-\sum_{h\in\mathscr{H}}\widetilde{\Lambda}_{Y_rh}\widehat{w}_h+\sum_{j\in\mathscr{N}_r}\widetilde{\lambda}_{Y_rj}\left(\widehat{\mathscr{A}}_j+\widehat{\mu}_j\right)-\sum_{i\notin\mathscr{N}_r}\left(\Omega_{Y_ri}+\widetilde{\Lambda}_{Y_ri}\right)\left(\widehat{\Omega}_{Y_ri}-\widehat{q}_{ri}+\widehat{GDP}_r\right)\\
&=\sum_{i\in\mathscr{N}_r}\sum_{h\in\mathscr{H}}\frac{J_{ih}}{GDP_r}\widehat{J}_{ih}+\sum_{i\in\mathscr{N}_r}\lambda_{Y_ri}\left(\left(1-\mu_i\right)\widehat{S}_i-\mu_i\widehat{\mu}_i\right)-\sum_{h\in\mathscr{H}}\widetilde{\Lambda}_{Y_rh}\widehat{J}_h+\sum_{h\in\mathscr{H}}\widetilde{\Lambda}_{Y_rh}\widehat{L}_h\\
&+\sum_{j\in\mathscr{N}_r}\widetilde{\lambda}_{Y_rj}\left(\widehat{\mathscr{A}}_j+\widehat{\mu}_j\right)-\sum_{i\notin\mathscr{N}_r}\left(\Omega_{Y_ri}+\widetilde{\Lambda}_{Y_ri}\right)\left(\widehat{\Omega}_{Y_ri}-\widehat{q}_{ri}+\widehat{GDP}_r\right)\\
&=\sum_{h\in\mathscr{H}}\Lambda_{Y_rh}\widehat{\Lambda}_{Y_rh}+\sum_{i\in\mathscr{N}_r}\lambda_{Y_ri}\left(\left(1-\mu_i\right)\widehat{\lambda}_{Y_ri}-\mu_i\widehat{\mu}_i\right)-\sum_{h\in\mathscr{H}}\widetilde{\Lambda}_{Y_rh}\widehat{J}_h+\sum_{h\in\mathscr{H}}\widetilde{\Lambda}_{Y_rh}\widehat{L}_h\\
&+\sum_{j\in\mathscr{N}_r}\widetilde{\lambda}_{Y_rj}\left(\widehat{\mathscr{A}}_j+\widehat{\mu}_j\right)+\sum_{i\notin\mathscr{N}_r}\left(\Omega_{Y_ri}+\widetilde{\Lambda}_{Y_ri}\right)\left(\widehat{q}_{ri}-\widehat{\Omega}_{Y_ri}\right)\\
&+\left(\sum_{h\in\mathscr{H}}\Lambda_{Y_rh}+\sum_{i\in\mathscr{N}_r}\left(1-\mu_i\right)\lambda_{Y_ri}-\sum_{i\notin\mathscr{N}_r}\left(\Omega_{Y_ri}+\sum_{j\in\mathscr{N}_r}\widetilde{\lambda}_{Y_rj}\widetilde{\Omega}^{x}_{jf}\right)\right)\widehat{GDP}_r.
\end{aligned}
$$

Let me assume as in their paper that factor markets are segmented by country, this implies that $\Lambda_{Y_rh}=\mathbb{1}\left\{h\in\mathscr{H}_r\right\}\frac{J_h}{GDP_r}$ and from equation (76)

- $\sum_{h\in\mathscr{H}_r}\Lambda_{Y_rh}+\sum_{i\in\mathscr{N}_r}\left(1-\mu_i\right)\lambda_{Y_ri}=1$; and
- $\sum_{h\in\mathscr{H}_r}\Lambda_{Y_rh}\widehat{\Lambda}_{Y_rh}+\sum_{i\in\mathscr{N}_r}\lambda_{Y_ri}\left(\left(1-\mu_i\right)\widehat{\lambda}_{Y_ri}-\mu_i\widehat{\mu}_i\right)=0.$

Therefore

$$
\begin{aligned}
\widehat{Y}_r=&\sum_{j\in\mathscr{N}_r}\widetilde{\lambda}_{Y_rj}\left(\widehat{\mathscr{A}}_j+\widehat{\mu}_j\right)-\sum_{h\in\mathscr{H}_r}\widetilde{\Lambda}_{Y_rh}\widehat{\Lambda}_{Y_rh}+\sum_{h\in\mathscr{H}_r}\widetilde{\Lambda}_{Y_rh}\widehat{L}_h+\sum_{i\notin\mathscr{N}_r}\left(\Omega_{Y_ri}+\widetilde{\Lambda}_{Y_ri}\right)\left(\widehat{q}_{ri}-\widehat{\Omega}_{Y_ri}\right)\\
&+\left(1-\sum_{i\notin\mathscr{N}_r}\Omega_{Y_ri}-\sum_{i\notin\mathscr{N}_r}\sum_{j\in\mathscr{N}_r}\widetilde{\lambda}_{Y_rj}\widetilde{\Omega}^{x}_{ji}-\sum_{h\in\mathscr{H}_r}\widetilde{\Lambda}_{Y_rh}\right)\widehat{GDP}_r.
\end{aligned}
$$

Notice that

$$
\begin{aligned}
&1-\sum_{i\notin\mathscr{N}_r}\Omega_{Y_r i}-\sum_{i\notin\mathscr{N}_r}\sum_{j\in\mathscr{N}_r}\widetilde{\lambda}_{Y_r j}\widetilde{\Omega}^x_{ji}-\sum_{h\in\mathscr{H}_r}\widetilde{\Lambda}_{Y_r h}\\
=&1-\sum_{i\notin\mathscr{N}_r}\Omega_{Y_r i}-\sum_{i\in\mathscr{N}_r}\Omega_{Y_r i}\sum_{j\in\mathscr{N}_r}\widetilde{\psi}^{x_r}_{ij}\left(\sum_{h\in\mathscr{H}_r}\widetilde{\Omega}^{\ell}_{jh}+\sum_{i\notin\mathscr{N}_r}\widetilde{\Omega}^x_{ji}\right)\\
=&1-\sum_{i\notin\mathscr{N}_r}\Omega_{Y_r i}-\sum_{i\in\mathscr{N}_r}\Omega_{Y_r f}\sum_{j\in\mathscr{N}_r}\widetilde{\psi}^{x_r}_{ij}\left(1-\sum_{i\in\mathscr{N}_r}\widetilde{\Omega}^x_{ji}\right)\\
=&1-\sum_{i\in\mathscr{N}_r}\Omega_{Y_r i}-\sum_{i\notin\mathscr{N}_r}\Omega_{Y_r i}\\
=&1-\frac{1}{GDP_r}\left(\sum_{i\in\mathscr{N}_r}\left(S_i-\sum_{j\in\mathscr{N}_r}p_i x_{ji}\right)-\sum_{f\notin\mathscr{N}_r}\sum_{j\in\mathscr{N}_r}p_i x_{ji}\right)\\
=&1-\frac{1}{GDP_r}\sum_{i\in\mathscr{N}_r}\left(S_i-\sum_{j\in\mathscr{N}}p_j x_{ij}\right)=0,
\end{aligned}
$$

where the third line is coming from the fact that $\left(I_{N_r}-\widetilde{\Omega}^{D_r}_x\right)^{-1}=I_{N_r}+\left(I_{F_r}-\widetilde{\Omega}^{D_r}_x\right)^{-1}\widetilde{\Omega}^{D_r}_x$.

Finally, define the value of relative imports from foreign firm i to domestic GDP as $\Lambda_{Y_r i}=-\frac{p_i q_{ri}}{GDP_r}$ for $i\in\mathscr{N}-\mathscr{N}_r$

$$
\widehat{Y}_r=\sum_{j\in\mathscr{N}_r}\widetilde{\lambda}_{Y_r j}\left(\widehat{\mathscr{A}}_j+\widehat{\mu}_j\right)-\sum_{h\in\mathscr{H}_r}\widetilde{\Lambda}_{Y_r h}\widehat{\Lambda}_{Y_r h}+\sum_{h\in\mathscr{H}_r}\widetilde{\Lambda}_{Y_r h}\widehat{L}_h+\sum_{i\notin\mathscr{N}_r}\left(\widetilde{\Lambda}_{Y_r i}-\Lambda_{Y_r i}\right)\left(\widehat{q}_{ri}-\widehat{\Lambda}_{Y_r i}\right), \tag{114}
$$

which is the first-order approximation for country-level GDP variation from Theorem 1 in Baqaee & Farhi (2023). There are still one differences between their result and equation (114). They define $\widetilde{\Lambda}_{Y_r i}=\sum_{j\in\mathscr{N}_r}\Omega_{Y_r j}\widetilde{\psi}_{ji}$ for $i\notin\mathscr{N}_r$ in the main text, instead of using $\widetilde{\Lambda}_{Y_r i}=\sum_{m\in\mathscr{N}_r}\Omega_{Y_r m}\sum_{j\in\mathscr{N}_r}\widetilde{\psi}^{x_r}_{mj}\widetilde{\Omega}^x_{ji}$ as in their appendix. I consider this problematic because it does not consider the difference between $\widetilde{\Psi}_x$ and $\left(I_{N_r}-\widetilde{\Omega}^{D_r}_x\right)^{-1}$, and because it leaves aside the role of $\widetilde{\Omega}^{M_r}_x$ in the definition of $\widehat{p}_{Y_r}$.

The differences between my first order approximation in equation (109) and Baqaee & Farhi's (2023) first order approximation in equation (114) are

1. Equation (109) characterizes the effect from productivity and markdown shocks from all firms, while equation (114) only characterizes the effect for firms that operate within country r;
2. For equation (109) it is not necessary to trace any variation for the real allocation of goods between countries, while for equation (114) it is necessary to trace the variation in the net quantity of goods imported to country r and their share with respect to domestic GDP, i.e. $\widehat{q}_{ri}$ and $\widehat{\Lambda}_{Y_r i}$ for $i\notin\mathscr{N}_r$;

3. Equation (109) does not require any segmentation of factor markets at the country level, while equation (114) does.

3.5 First-order approximation for household-level Real Consumption

Equation (101) captures the first-order approximation for real consumption at the household level. Theorem 2 in Baqaee & Farhi (2023) brings a comparable approximation for welfare variation at the household level that is given by[6]

$$\widehat{C}_h = \sum_{i\in\mathscr{N}} \widetilde{\lambda}_{C_h i}\left(\widehat{A}_i + \widehat{\mu}_i\right) + \sum_{b\in\mathscr{L}\cup\mathscr{L}^*}\left(\Lambda_{C_h b} - \widetilde{\Lambda}_{C_h b}\right)\widehat{\Lambda}_b + \sum_{b\in\mathscr{L}} \widetilde{\Lambda}_{C_h b}\widehat{L}_b.$$

This result matches equation (101) once we consider that

- $\widetilde{\lambda}_{C_h i} = \widetilde{\mathscr{B}}_{hi}$;
- $\widetilde{\Lambda}_{C_h b} = \widetilde{\mathscr{C}}_{hb}$;
- $\sum_{b\in\mathscr{L}\cup\mathscr{L}^*}\left(\Lambda_{C_h b} - \widetilde{\Lambda}_{C_h b}\right)\widehat{\Lambda}_b = \widehat{\chi}_h - \sum_{b\in\mathscr{H}} \widetilde{\mathscr{C}}_{hb}\widehat{\Lambda}_b$.

[6]As primary factor supply in Baqaee & Farhi (2023) is inelastic, household real consumption variation is identical to welfare variation.